CORRESPONDENCE THEOREMS FOR TROPICAL CURVES

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Abstract. In this paper, we study the correspondence between tropical curves and holomorphic curves. Previously this was studied in the case of so-called non-superabundant tropical curves by Mikhalkin, and by Siebert and the author. The main subjects in this paper are superabundant tropical curves. First we give an effective combinatorial description of these curves. Based on this description, we calculate the obstructions for appropriate deformation theory, describe the Kuranishi map, and study the space of solutions of it. The genus one case is solved completely, and the theory works for many of the higher genus cases, too.

1. Introduction

In this paper, we consider the correspondence between tropical curves in real affine spaces and holomorphic curves in toric varieties. This study was initiated by G.Mikhalkin's celebrated paper [4], in which he proved the correspondence between tropical curves of any genus in $\mathbb{R}^2$, and holomorphic curves in toric surfaces specified by combinatorial data of the tropical curves. Subsequently, B. Siebert and the author proved the correspondence between rational tropical curves in $\mathbb{R}^n$ and rational curves in n-dimensional toric varieties [9]. We extend these results to the correspondence between tropical curves of any genus in $\mathbb{R}^n$ and holomorphic curves in n-dimensional toric varieties.

Our first result is the unification and the extension of the above two results. Namely, the correspondence theorem for general non-superabundant tropical curves. This result was first announced by Mikhalkin in his paper [5, Theorem 1]. The terminologies used in the statement are defined or explained in Section 2.

Theorem 1. *Let (Γ, h), $h : \Gamma \to \mathbb{R}^n$ be an immersive tropical curve of genus g which is non-superabundant. Let X be an n-dimensional toric variety associated to (Γ, h) and $\mathfrak{X} \to \mathbb{C}$ be a degeneration of X defined respecting (Γ, h). Let X_0 be the central fiber of $\mathfrak{X}$. Then any maximally degenerate pre-log curve in X_0 of type (Γ, h) can be smoothly deformed*

1991 *Mathematics Subject Classification.* 14T05 (primary), 14N35, 14M25 (secondary).

Key words and phrases. Tropical curves, Gromov-Witten invariants, Toric varieties.

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into a holomorphic curve in X, and the degrees of freedom of deforming tropical and holomorphic curves coincide.

Using the terminology in Definition 23, this theorem can be stated in shorter terms. Namely, it is equivalent to the following statement: For an immersive non-superabundant tropical curve (Γ, h), any pre-log curve of type (Γ, h) is smoothable.

Remark 2. *From this theorem, we can deduce enumerative results for non-superabundant curves as in* [9]*, introducing incidence conditions, markings of the edges, various weights, etc.. We leave the precise formulation to the interested readers, because it can be performed completely similarly as in* [9]*. We develop the (more involved) study of enumeration problems of genus one case in Subsection 6.3.*

However, this is not our main result. In fact, the proof of Theorem 1 is more important than the result itself, for our purpose. In the proof of Proposition 25, we develop a new combinatorial method to describe sheaf cohomology groups of holomorphic curves associated to tropical curves. Using this idea, we obtain an effective combinatorial description of superabundant tropical curves (Theorem 30). This is the starting point of our study of these curves.

Then we begin to study the deformation theory. Our general strategy to prove the correspondence theorem is as follows: First we construct a singular curve (pre-log curve, Definition 19) in a singular variety (central fiber of a toric degeneration), then we try to find the necessary and sufficient condition under which the singular curve deforms to a smooth curve.

The main difficulty is the existence of the obstructions for the deformation, which was absent in the non-superabundant case. In Section 5, we calculate these obstructions for genus one case (Proposition 41). The description of superabundant curves (Theorem 30) turns out to be very well fitted to this calculation.

Based on this calculation, we obtain the necessary and sufficient condition for a genus one superabundant tropical curve (Γ, h) under which there is a pre-log curve associated to (Γ, h) which allows a deformation (Theorems 45, 52). The result turns out to be the (extension of) *well-spacedness condition* introduced by D.Speyer [12].

However, these 'existence' theorems are not enough for the 'correspondence' theorem. In particular, we cannot deduce enumerative correspondences between tropical and holomorphic curves from these existence theorems. In Section 6, we develop correspondences between moduli spaces of tropical and holomorphic curves. We study the 'microscopic' role of the tropical curves (Remark 60), which determines the moduli of the pre-log curves (contrary to the 'macroscopic' role of them which determines the toric degeneration).

Another important point is that we need to consider tropical curves which need not be immersions. The detailed study of this type of tropical curves begins with Subsection 5.2.

Then we define the Kuranishi map (Definition 53), whose zero set is the moduli space of pre-log curves which allow deformations. In genus one case, we can study the zero set of the Kuranishi map in detail, and based on this, we prove the correspondence between moduli spaces (Theorems 62, 63), and the enumerative correspondence theorem (Theorem 71) follows as a corollary.

In Section 7, we study the superabundant tropical curves of higher genus. We can define the Kuranishi map also in these general cases, but the study of the zero set of it in general situation becomes difficult. However, in some cases, for example in low genus cases or in the cases where the tropical curve has only one connected component of loops (Definition 12), one can still study the Kuranishi map. We give several examples of these cases.

It is possible to extend the results of this paper to several directions. In [7], we developed an algebraic method to deal with stable discs in toric varieties whose boundary is mapped to torus fibers of the moment map. Combining it with the method of this paper, it is possible to study general bordered Riemannian surfaces in toric varieties whose boundary components are mapped to torus fibers. In [10], the technic of toric degeneration is applied to calculate superpotentials of torus fibers of Gelfand-Cetlin integrable systems on flag varieties of type A. The same technic, as well as the method in [7] was applied in [11], to study other types of toric degeneration, including $\mathbb{P}^1 \times \mathbb{P}^1$ and cubic surfaces. Also, in [8], we developed a method to study Gromov-Witten type invariants of genus zero of some Fano manifolds by tropical geometry. Using the method of this paper, all these considerations for disks or genus zero curves can be extended to the study of general bordered Riemannian surfaces.

Assumptions made in this paper. In this paper, there are three assumptions made, Assumption A (Subsection 2.1), B (Section 3), and C (Subsection 6.1). The relation between them is

$$\text{Assumption A} < \text{Assumption C} < \text{Assumption B},$$

where $P < Q$ means P is a weaker assumption than Q. In fact, almost all of the arguments can be extended to Assumption A. However, tropical curves satisfying only Assumption A can have arbitrary complexity in the non-loop part, and this makes it difficult to describe the statements of the theorems and their proofs in a uniform way. So the actual arguments will be given under moderately stronger assumptions. The strength of the assumptions is determined by the degrees of generality under which the enumerative results can be proved.

The strongest Assumption B is adopted only in Section 3, where we consider non-superabundant tropical curves. In Section 4, the argument is given again under Assumption B, however, it is easy to see that the result applies to all the cases satisfying Assumption A, since the non-loop part plays little role in this section (see Remark 26). From Section 5 to Section 7, we work mostly under Assumption C, which is enough for the genus one enumerative result. Most of the arguments there can be extended to the curves satisfying only Assumption A, however, for the enumeration problems in higher genus cases, we have to work under even weaker conditions than Assumption A (see Remark 33, Example 50). So we leave these extensions to future study.

Acknowledgments. Needless to say, I was inspired by D.Speyer's paper [12]. My study of tropical geometry began from the joint work [9] with B.Siebert, and many ideas from it appear in this paper, too. It is a great pleasure for me to express my gratitude to him. I am supported by Grant-in-Aid for Young Scientists (No. 19740034).

2. Preliminaries

In this section, we recall and define some notations and notions which are used in this paper.

2.1. **Tropical curves.** First we recall some definitions about tropical curves, see [4, 9] for more information. Let $\overline{\Gamma}$ be a weighted, connected finite graph. Its sets of vertices and edges are denoted $\overline{\Gamma}^{[0]}$, $\overline{\Gamma}^{[1]}$, and $w_{\overline{\Gamma}} : \overline{\Gamma}^{[1]} \to \mathbb{N}\backslash\{0\}$ is the weight function. An edge $E \in \overline{\Gamma}^{[1]}$ has adjacent vertices $\partial E = \{V_1, V_2\}$. Let $\overline{\Gamma}^{[0]}_\infty \subset \overline{\Gamma}^{[0]}$ be the set of one-valent vertices. We write $\Gamma = \overline{\Gamma} \setminus \overline{\Gamma}^{[0]}_\infty$. Noncompact edges of Γ are called *unbounded edges*. Let $\Gamma^{[1]}_\infty$ be the set of unbounded edges. Let $\Gamma^{[0]}, \Gamma^{[1]}, w_\Gamma$ be the sets of vertices and edges of Γ and the weight function of Γ (induced from $w_{\overline{\Gamma}}$ in an obvious way), respectively. Let N be a free abelian group of rank $n \geq 2$ and $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$.

Definition 3 ([4, Definition 2.2])**.** A *parametrized tropical curve* in $N_\mathbb{R}$ is a proper map $h : \Gamma \to N_\mathbb{R}$ satisfying the following conditions.

(i) For every edge, $E \subset \Gamma$ the restriction $h\big|_E$ is an embedding with the image $h(E)$ contained in an affine line with rational slope, or $h(E)$ is a point.

(ii) For every vertex $V \in \Gamma^{[0]}$, $h(V) \in N_\mathbb{Q}$ and the following *balancing condition* holds. Let $E_1, \dots, E_m \in \Gamma^{[1]}$ be the edges adjacent to V and let $u_i \in N$ be the primitive integral vector emanating from $h(V)$ in the direction of $h(E_i)$. Then

$$\sum_{j=1}^{m} w(E_j)u_j = 0. \tag{1}$$

Remark 4. *In* [9], $h|_E$ *is assumed to be an embedding (see* [9, Definition 1.1]*) for every edge* E*. The reason that we take the above definition is that those cases appear naturally when we consider superabundant tropical curves. See Assumption A and the paragraph before it.*

An isomorphism of parametrized tropical curves $h : \Gamma \to N_{\mathbb{R}}$ and $h' : \Gamma' \to N_{\mathbb{R}}$ is a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the weights such that $h = h' \circ \Phi$.

Definition 5. A *tropical curve* is an isomorphism class of parametrized tropical curves. A tropical curve is *trivalent* if Γ is a trivalent graph. The *genus* of a tropical curve is the first Betti number of Γ. The set of *flags* of Γ is

$$F\Gamma = \{(V, E) \big| V \in \partial E\}.$$

By (i) of Definition 3, we have a map $u : F\Gamma \to N$ sending a flag (V, E) to the primitive integral vector $u_{(V,E)} \in N$ emanating from $h(V)$ in the direction of $h(E)$.

Definition 6. The (unmarked) *combinatorial type* of a tropical curve (Γ, h) is the graph Γ together with the map $u : F\Gamma \to N$. We write this by the pair (Γ, u).

Definition 7. For $l \in \mathbb{N}$, an *l-marked* tropical curve is a tropical curve (Γ, h) together with a choice of l edges $\mathbf{E} = (E_1, \dots, E_l) \subset (\Gamma^{[1]})^l$. We write the l-marked tropical curves as $(\Gamma, \mathbf{E}, h)$. The elements of $\{E_i\}$ need not be pairwise distinct. For an l-marked tropical curve, we define the (marked) *combinatorial type* by the data of the marking $\mathbf{E}$ of the graph Γ together with the (unmarked) combinatorial type (Γ, u).

Definition 8. The *degree* of a type (Γ, u) is a function $\Delta : N\setminus\{0\} \to \mathbb{N}$ with finite support defined by

$$\Delta(\Gamma, u)(v) := \sharp\{(V, E) \in F\Gamma | E \in \Gamma^{[1]}_{\infty}, w(E)u_{(V,E)} = v\}$$

Let $e = |\Delta| = \sum_{v \in N\setminus\{0\}} \Delta(v)$. This is the same as the number of unbounded edges of Γ (not necessarily of $h(\Gamma)$).

Proposition 9 ([4, Proposition 2.13])**.** *The moduli space of trivalent tropical curves of given combinatorial type is, if it is non-empty, an open convex polyhedral domain in a real affine k-dimensional space, where $k \geq e + (n-3)(1-g)$.* □

Definition 10 ([4, Definition 2.22])**.** A trivalent tropical curve is called *superabundant* if the moduli space is of dimension larger than $e + (n-3)(1-g)$.

Definition 11. We call a tropical curve (Γ, h) *immersive* if h is an immersion and if $V \in \Gamma^{[0]}$, then $h^{-1}(h(V)) = \{V\}$.

In the case of genus zero ([9]), or more generally, in the non-superabundant case, one can see that if (Γ, h) is a tropical curve which satisfies a generic constraint $\mathbf{A} = (A_1, \dots, A_l)$ of codimension $\mathbf{d} = (d_1, \dots, d_l)$ (see [9, Definition 2.3], or Subsection 6.3 of this paper) with $|\mathbf{d}| = \sum_i d_i = (n-3)(1-g) + e$, then h is an immersion (embedding if n is greater than two). In particular:

- The image $h(\Gamma)$ is also a trivalent graph when $n \geq 3$.
- No edge of Γ is contracted.
- The weights of the corresponding edges of Γ and $h(\Gamma)$ are the same.

However, in the superabundant case, the situation that h contracts some of the edges of Γ naturally appears, and consequently, the image curve $h(\Gamma)$ may have vertices of higher valence (> 3), and some of the edges of Γ may have the same image. On the other hand, we do not need to allow all kind of h for the enumerative results (see Remark 33). In this paper, we assume that a tropical curve (Γ, h) satisfies the following Assumption A.

To state Assumption A, we prepare some terminologies. Let Γ be a finite graph as above.

Definition 12. (i) An edge $E \in \Gamma^{[1]}$ is said to be a *part of a loop of* Γ if the graph given by $\Gamma \setminus E^\circ$ has lower first Betti number than Γ. Here E° is the interior of E (that is, $E^\circ = E \setminus \partial E$).

(ii) The *loops of* Γ is the subgraph of Γ composed by the union of parts of a loop of Γ.

(iii) A *bouquet of* Γ is a connected component of the loops of Γ. If the first Betti number of a bouquet is one, it is called a *loop*.

In particular, a bouquet or a loop does not contain unbounded edges. Now we state Assumption A.

Assumption A.

(i) The abstract graph Γ is always trivalent. So (Γ, h) is always a trivalent tropical curve in the above terminology, although the image $h(\Gamma)$ may not be trivalent.

(ii) The map h may contract some of the bounded edges of Γ. However, a contracted edge does not have an intersection with the loops of Γ (including the ends of the edge).

(iii) Some of the vertices of Γ may have the same image in $h(\Gamma)$. Assume $p, q \in \Gamma^{[0]}$ have the same image in $h(\Gamma)$. Then p and q are connected by a path of edges in Γ which are contracted by h.

(iv) When $n \geq 3$ (in particular, when (Γ, h) is superabundant), if $E \in \Gamma^{[1]}$ is not contracted by h, then $h(E^\circ) \cap h(\Gamma \setminus E^\circ)$ is an empty set.

Under this assumption, one easily deduces the following properties.

Lemma 13. *Under Assumption A, the following properties of* (Γ, h) *hold.*

(i) If $v \in \Gamma^{[0]}$ *is a vertex contained in the loops of* Γ*, then the image* $h(v)$ *is also trivalent.*

(ii) Let E_1, E_2 *be edges of* Γ *which are not contracted. Assume the relation* $h(E_1) \subset h(E_2)$ *holds. Then* E_1 *and* E_2 *are unbounded edges of* Γ*, and* $h(E_1) = h(E_2)$*.* □

In order to distinguish the valence and the weights between Γ and $h(\Gamma)$, we introduce the following definitions.

Definition 14. We assume $n \geq 3$ and Assumption A.

(i) A *vertex* of $h(\Gamma)$ is the image of some vertex of Γ.

(ii) Let $\mathfrak{V} \in h(\Gamma)$ be a vertex and let $V_1, \dots, V_s \in \Gamma^{[0]}$ be all the vertices of Γ whose image is $\mathfrak{V}$. The *valence* of $\mathfrak{V}$, $val(\mathfrak{V})$ is defined as follows. Namely, by Assumption A, $V_1, \dots, V_s$ are connected by edges of Γ which are contracted by h. Contracting these edges in Γ produces a graph with a vertex W which is the image of $V_1, \dots, V_s$ under this contraction. Then,
$$val(\mathfrak{V}) = \text{valence of } W.$$
This equals $s + 2$ because Γ is trivalent.

(iii) Let $\mathfrak{E} \in h(\Gamma)$ be an edge. Let $E_1, \dots, E_s \in \Gamma^{[1]}$ be the edges of Γ such that $h(E_i) = \mathfrak{E}$ (in particular, we assume the edges $E_1, \dots, E_s$ are not contracted by h). Then the *weight* of $\mathfrak{E}$, $w(\mathfrak{E})$, is the (unordered) set of positive integers $\{w_1, \dots, w_s\}$, here w_i is the weight of E_i in Γ. The sum $w_s(\mathfrak{E}) = \sum_{i=1}^{s} w_i$ is called the *total additive weight* of $\mathfrak{E}$. The product $w_m(\mathfrak{E}) = \prod_{i=1}^{s} w_i$ is called the *total weight* of $\mathfrak{E}$.

Definition 15. Let $h(\Gamma)^{[1]}$ be the set of the edges of $h(\Gamma)$ and $h(\Gamma_\infty^{[1]})$ be the set of the unbounded edges of $h(\Gamma)$. The *total inner weight* $w(\Gamma, h)$ of a tropical curve (Γ, h) is the product
$$w(\Gamma, h) = \prod_{\mathfrak{E} \in h(\Gamma)^{[1]} \setminus h(\Gamma_\infty^{[1]})} w_m(\mathfrak{E}).$$

Note that this is not equal to the total inner weight defined in [9, Section 1]. Namely, in the definition here, the weights of the contracted edges do not contribute to the total inner weight. In [9], we could assume that all the tropical curves were immersive, so this point did not appear.

Definition 16. For an l-marked tropical curve $(\Gamma, \mathbf{E}, h)$, the *total marked weight* is the product
$$w(\Gamma, \mathbf{E}, h) = w(\Gamma, h) \cdot \prod_{i=1}^{l} w(E_i).$$

Note that in the second factor, the weight is taken in the abstract graph Γ, not in the image $h(\Gamma)$.

2.2. **Toric varieties associated to tropical curves and pre-log curves on them.**

Definition 17. A toric variety X defined by a fan Σ is called to be *associated to a tropical curve* (Γ, h) if the set of the rays of Σ contains the set of the rays spanned by the vectors in N which are contained in the support of the degree map $\Delta : N \setminus \{0\} \to \mathbb{N}$ of (Γ, h).

If $\mathfrak{E}$ is an unbounded edge of $h(\Gamma)$, there is an obvious unique divisor of X corresponding to it. We write it as $D_{\mathfrak{E}}$ and call it the *divisor associated to the edge* $\mathfrak{E}$.

Given a tropical curve (Γ, h) in $N_{\mathbb{R}}$, we can construct a polyhedral decomposition $\mathscr{P}$ of $N_{\mathbb{R}}$ such that $h(\Gamma)$ is contained in the 1-skeleton of $\mathscr{P}$ ([9, Proposition 3.9]). Given such $\mathscr{P}$, we construct a degenerating family $\mathfrak{X} \to \mathbb{C}$ of a toric variety X associated to (Γ, h) ([9, Section 3]). We call such a family a *degeneration of* X *defined respecting* (Γ, h). Let X_0 be the central fiber. It is a union $X_0 = \cup_{v \in \mathscr{P}^{[0]}} X_v$ of toric varieties intersecting along toric strata. Here $\mathscr{P}^{[0]}$ is the set of the vertices of $\mathscr{P}$.

Definition 18 ([9, Definition 4.1])**.** Let X be a toric variety. A holomorphic curve $C \subset X$ is *torically transverse* if it is disjoint from all toric strata of codimension greater than one. A stable map $\phi : C \to X$ is torically transverse if $\phi^{-1}(intX) \subset C$ is dense and $\phi(C) \subset X$ is a torically transverse curve. Here $intX$ is the complement of the union of toric divisors.

Definition 19. Let C_0 be a prestable curve. A *pre-log curve* on X_0 is a stable map $\varphi_0 : C_0 \to X_0$ with the following properties.

(i) For any v, the restriction $C \times_{X_0} X_v \to X_v$ is a torically transverse stable map.
(ii) Let $P \in C_0$ be a point which maps to the singular locus of X_0. Then C has a node at P, and φ_0 maps the two branches $(C_0', P), (C_0'', P)$ of C_0 at P to different irreducible components $X_{v'}, X_{v''} \subset X_0$. Moreover, if w' is the intersection index of the restriction $(C_0', P) \to (X_{v'}, D')$ with the toric divisor $D' \subset X_{v'}$, and w'' accordingly for $(C_0'', P) \to (X_{v''}, D'')$, then $w' = w''$.

Let X be a toric variety and D be the union of toric divisors. In [9, Definition 5.2], a non-constant torically transverse map $\phi : \mathbb{P}^1 \to X$ is called a *line* if $\sharp\phi^{-1}(D) \leq 3$. Because we consider more general tropical curves, we have to extend this notion.

Let Γ be a tree and $h : \Gamma \to N_{\mathbb{R}}$ be a tropical curve. Assume $h(\Gamma)$ has only one vertex. Let $\mathfrak{E}_1, \dots, \mathfrak{E}_s$ be the edges of $h(\Gamma)$. Let X be a toric variety associated to (Γ, h).

Definition 20. A non-constant torically transverse map $\phi : \mathbb{P}^1 \to X$ is called *of type* (Γ, h) (or, if $\mathfrak{V} \in h(\Gamma)$ is the unique vertex, we may call *of type* $\mathfrak{V}$ when no confusion will occur) if ϕ satisfies the following properties.

(i) Let $\mathfrak{E}_i$ be an edge of $h(\Gamma)$ and let $w(\mathfrak{E}_i) = \{w_{i,1}, \dots, w_{i,m_i}\}$ be the weight of $\mathfrak{E}_i$. Then $\phi(\mathbb{P}^1)$ intersects $D_{\mathfrak{E}_i}$ at m_i different points.
(ii) Their intersection multiplicity is given by $\{w_{i,j}\}_{j=1,\dots,m_i}$ (we do not specify the order).

Under Assumption A, if $\mathfrak{V}$ is a part of a larger tropical curve (Γ', h'), then $m_i > 1$ occurs only if $\mathfrak{E}_i$ is an unbounded edge.

Let (Γ, h) be a tropical curve satisfying Assumption A. Let X be a toric variety associated to (Γ, h) and $\mathfrak{X} \to \mathbb{C}$ be a degeneration of X defined respecting (Γ, h). Let X_0 be the central fiber.

Definition 21. Let us assume $n \geq 3$. A pre-log curve $\varphi_0 : C_0 \to X_0$ is called *of type* (Γ, h) if for any $\mathfrak{V} \in h(\Gamma^{[0]}) \subset \mathscr{P}^{[0]}$, the restriction $C_0 \times_{X_0} X_{\mathfrak{V}} \to X_{\mathfrak{V}}$ is a rational curve of type $\mathfrak{V}$.

Remark 22. *If* (Γ, h) *is immersive, then a pre-log curve of type* (Γ, h) *is just the maximally degenerate curve of* [9, Definition 5.6].

Definition 23. A tropical curve (Γ, h) satisfying Assumption A is *smoothable* if the following holds: There is a pre-log curve $\varphi_0 : C_0 \to X_0$ of type (Γ, h) with the following property. Namely, there exists a family of stable maps over a pointed curve (D, x_0)

$$\Phi : \mathfrak{C}/D \to \mathfrak{X}/D$$

such that $\mathfrak{C}/D$ is a flat family of pre-stable curves whose fiber over x_0 is isomorphic to C_0, and the restriction of Φ to x_0 is a stable map equivalent to φ_0. We also call such a pre-log curve *smoothable.*

Remark 24. *The smoothability of a tropical curve does not depend on the choice of a toric variety* X *associated to it or a degeneration of* X *defined respecting the tropical curve.*

See [9, Section 5], for more information about lines and maximally degenerate pre-log curves. Given an immersive trivalent tropical curve, we can construct maximally degenerate pre-log curves ([9, Proposition 5.7]), and vice versa ([9, Construction 4.4]). The arguments there extend to not necessarily immersive trivalent tropical curves and pre-log curves of type (Γ, h), if (Γ, h) satisfies Assumption A. We give some details for the cases relevant to the enumeration problem in Subsection 5.2.

The smoothings of the maximally degenerate curves or pre-log curves of type (Γ, h) are given by log-smooth deformation theory [2, 3]. For informations about log structures relevant to our situation, see [9, Section 7]. We do not repeat it here, because nothing new about log structures is required here, other than those given in [9].

3. Proof of non-superabundant correspondence theorem

The purpose of this section is to give a proof of Theorem 1.

Assumption B. In this section, we assume that if (Γ, h) is a tropical curve, then h is an immersion (an embedding when $n \geq 3$), because this suffices for the (generic) enumeration problem for non-superabundant tropical curves (see the paragraph before Definition 12).

Let $\varphi_0 : C_0 \to X_0$ be a maximally degenerate pre-log curve of type (Γ, h), a given trivalent non-superabundant tropical curve as in the previous section. We can give it log-structures as in [9, Proposition 7.1]. We assume that a lift $\varphi_{k-1} : C_{k-1}/O_{k-1} \to \mathfrak{X}$ of φ_0 is constructed. Here $O_{k-1} = \mathbb{C}[\epsilon]/\epsilon^k$. Then as in the proof of [9, Lemma 7.2], an extension C_k/O_k of C_{k-1}/O_{k-1} exists and such extensions are parametrized by the space of extensions of appropriate sheaves.

But the problem of lifting the map $\varphi_{k-1} : C_{k-1} \to \mathfrak{X}$ to C_k is different from [9], due to the existence of obstructions. The obstruction is given by the cohomology class $H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})$, here $\Theta_{\mathfrak{X}/\mathbb{C}}$ is the logarithmic tangent bundle relative to the base, and we will study this group in the following subsections.

3.1. Two dimensional case.

We begin, as a warm-up, with the two dimensional case which is easier and illustrates the problem to solve. Also, it will give a simple algebraic geometric proof of a part of Mikhalkin's correspondence theorem ([4, Theorem 1]).

The problem is the smoothing of maximally degenerate pre-log curves in the central fiber X_0 to a family of curves in $\mathfrak{X}$.

The sheaf $\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}$ fits in the exact sequence

$$0 \to \Theta_{C_{k-1}/O_{k-1}} \to \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}} \to \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}} \to 0. \tag{2}$$

Now $\Theta_{\mathfrak{X}/\mathbb{C}} \simeq N \otimes_{\mathbb{Z}} \mathcal{O}_{\mathfrak{X}}$ and the logarithmic tangent bundle $\Theta_{C_{k-1}/O_{k-1}}$ has degree $2-2g-e$, where e is the number of unbounded edges of the tropical curve from which we construct the pre-log curve (so that C_{k-1} has e marked points aside from the nodes). So the logarithmic normal bundle $\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}}$ has degree $2g+e-2$. Then by Serre duality for nodal curves, one can easily prove

$$H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}}) = 0$$

(this is the point where the assumption that X is of two dimension simplifies the argument).

So we have the surjection

$$H^1(C_{k-1}, \Theta_{C_{k-1}/O_{k-1}}) \to H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}).$$

However, $H^1(C_{k-1}, \Theta_{C_{k-1}/O_{k-1}})$ is just the tangent space of the moduli space of deformations of C_{k-1}, so the obstruction classes in $H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})$

can be cancelled when we deform the moduli of the domain of the stable maps. Thus, we can lift φ_{k-1} to φ_k also in this situation. Once the existence of a lift of the map is shown, the remainder of the proof of [9] applies verbatim, so this (and the results concerning weights and incidence conditions as Propositions 5.7, 7.1 and 7.3 of [9]) gives another proof of Mikhalkin's correspondence theorem for plane closed curves. □

The principle of the proof is the same for general higher dimensional cases, *in so far as the curve is non-superabundant*. Namely, we can show that the obstruction classes $H^1(C_k.\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})$ come only from the moduli of the curve itself. We will show this in the next subsection, which gives the proof of Theorem 1.

3.2. **General non-superabundant cases.** Let us consider the case when the rank of N is not less than three. We use the same notations as in the previous subsection. As before, we assume that a $(k-1)$-th lift φ_{k-1} of φ_0 has been constructed. The obstruction to lift φ_{k-1} to a k-th deformation is $H^1(C_{k-1},\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})$ as we noted. Consider the exact sequence (2) of Subsection 3.1 and the associated cohomology exact sequence. We have

$$\begin{aligned}&0\to H^0(C_{k-1},\Theta_{C_{k-1}/O_{k-1}})\to H^0(C_{k-1},\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})\to H^0(C_{k-1},\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})\\&\to H^1(C_{k-1},\Theta_{C_{k-1}/O_{k-1}})\to H^1(C_{k-1},\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})\to H^1(C_{k-1},\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})\to 0.\end{aligned}$$

The logarithmic tangent bundle $\Theta_{C_{k-1}/O_{k-1}}$ is, when it is restricted to each component of C_{k-1}, isomorphic to $\mathcal{O}_{C_{k-1}}(-1)$. So

$$H^0(C_{k-1},\Theta_{C_{k-1}/O_{k-1}})=0.$$

We have $\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}\cong\mathcal{O}_{C_{k-1}}\otimes_{\mathbb{Z}}N$. So

$$H^0(C_{k-1},\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})\cong\mathbb{C}[t]/t^k\otimes_{\mathbb{Z}}N.$$

The cohomology group $H^1(C_{k-1},\Theta_{C_{k-1}/O_{k-1}})$ is the tangent space of the moduli space of the curve C_{k-1} itself. By Serre duality for nodal curves, the space $H^1(C_{k-1},\Theta_{C_{k-1}/O_{k-1}})$ is isomorphic to the dual of the space $H^0(C_{k-1},\omega_{C_{k-1}}\otimes\Theta_{C_{k-1}/O_{k-1}}^\vee)$. Here $\omega_{C_{k-1}}$ is the dualizing sheaf, which is isomorphic to the sheaf of 1-forms with logarithmic poles at nodes. So when a component ℓ_i of C_{k-1} has s nodes,

$$\omega_{C_{k-1}}\otimes\Theta_{C_{k-1}/O_{k-1}}^\vee|_{\ell_i}\cong\mathcal{O}_{\ell_i}(-1+s).$$

To give a section of $H^0(C_{k-1},\omega_{C_{k-1}}\otimes\Theta_{C_{k-1}/O_{k-1}}^\vee)$, the value of the section on each component must coincide at the nodes. Now the rank (over $\mathbb{C}[t]/t^k$) of $H^0(C_{k-1},\omega_{C_{k-1}}\otimes\Theta_{C_{k-1}/O_{k-1}}^\vee)$ can be easily calculated as follows.

Consider the dual graph of C_{k-1}. Every vertex is trivalent (this is just the graph Γ since by Assumption B, h is an embedding when $rank\,N=n\geq 3$).

We first give every vertex three dimensional vector space $\mathbb{C}^3$, corresponding to $3 = \dim H^0(\mathbb{P}^1, \mathcal{O}(2))$, and consider the space

$$\prod_{v \in \Gamma^{[0]}} \mathbb{C}^3.$$

Then every unbounded edge (which does not contribute to s) as well as inner (in other words, bounded) edge imposes one dimensional linear conditions to the space $\prod_{v \in \Gamma^{[0]}} \mathbb{C}^3$. This is because an unbounded edge imposes zero of the section at the corresponding marked point, and a bounded edge imposes the matching of the values of the section at the node corresponding to the edge.

The resulting vector space has dimension

$$3v - e_{tot},$$

here v is the number of the vertices and e_{tot} is the number of the edges. We write

$$e_{tot} = e + e_{inn},$$

here e is the number of the unbounded edges and e_{inn} is the number of the bounded edges.

By Euler's equality, we have

$$1 - g = v - e_{inn}.$$

On the other hand, since the graph is trivalent,

$$e_{tot} = 3v - e_{inn}.$$

From these equalities, we have

$$3v - e_{tot} = e + 3g - 3.$$

So

$$\dim H^1(C_{k-1}, \Theta_{C_{k-1}/O_{k-1}}) = e + 3g - 3.$$

Now consider $H^1(C_{k-1}, \varphi_{k-1}^* \Theta_{\mathfrak{X}/\mathbb{C}})$. As above,

$$H^1(C_{k-1}, \varphi_{k-1}^* \Theta_{\mathfrak{X}/\mathbb{C}}) \simeq H^0(C_{k-1}, \omega_{C_{k-1}} \otimes N^\vee).$$

In this case, if a component ℓ_i of C_{k-1} has s nodes, then the restriction of $\omega_{C_{k-1}} \otimes N^\vee$ will be isomorphic to $\mathcal{O}_{\ell_i}(-2 + s) \otimes N^\vee$.

The next is the key to this section. The proof is important as well, because it plays a central role in the description of the superabundant curves in the next section.

Proposition 25. $\dim H^1(C_{k-1}, \varphi_{k-1}^* \Theta_{\mathfrak{X}/\mathbb{C}}) = ng$.

Proof. By Serre duality, it suffices to show $\dim H^0(C_{k-1}, \omega_{C_{k-1}}) = g$. Note that a trivalent tropical curve corresponds to a smooth rational curve $\mathbb{P}^1$ with three marked points.

Let z be an affine coordinate of $\mathbb{C} \subset \mathbb{P}^1$ and let a, b and c be distinct points on $\mathbb{P}^1$. Assume for simplicity that none of a, b, c is ∞. Let $\widetilde{\omega}$ be a sheaf of holomorphic 1-forms allowing logarithmic poles at a, b, c. Then

the space of sections $\Gamma(\widetilde{\omega})$ is a two dimensional vector space spanned by

$$\sigma = \frac{dz}{(z-a)(z-b)}, \quad \tau = \frac{dz}{(z-a)(z-c)}.$$

Taking

$$\frac{dz}{z-a}, \frac{dz}{z-b}, \frac{dz}{z-c}$$

as frames of $\widetilde{\omega}$ at a, b, c respectively, the section σ takes values

$$\frac{1}{a-b}, \frac{1}{b-a}, 0$$

respectively at a, b, c. Similarly, τ takes values

$$\frac{1}{a-c}, 0, \frac{1}{c-a}$$

respectively at a, b, c. In other words, the space of sections $\Gamma(\widetilde{\omega})$ is identified with the subspace of $\mathbb{R}^3 = \{(v_a, v_b, v_c) \mid v_a, v_b, v_c \in \mathbb{R}\}$ defined by

$$v_a + v_b + v_c = 0.$$

Using this convention, we reduce the problem to a combinatorial one. Consider a vertex v of the corresponding tropical curve Γ and let s be the number of bounded edges emanating from it as above.

(1) When $s = 1$, then the space of sections of $\mathcal{O}_{\ell_i}(-2+s)$ is trivial, and we give the value 0 to all the edges emanating from v.
(2) When $s = 2$, then we give 0 to the unbounded edge and give values $\pm a \in \mathbb{C}$ to the remaining two edges, respectively.
(3) When $s = 3$, we give values a, b, c satisfying $a + b + c = 0$ to the edges.

Thus, we give a number to each flag of Γ. We say that a numbering is *compatible* when the sum of the values of the two flags associated to a bounded edge is zero, reflecting the relation of the frames

$$\frac{dz_1}{z_1} + \frac{dz_2}{z_2} = 0$$

at a node, here z_1, z_2 are coordinates of the two branches at the node.

Then $\dim H^0(C_{k-1}, \omega_{C_{k-1}})$ is the number of linearly independent compatible numberings. So our task is reduced to the calculation of the number of linearly independent compatible numberings.

Now we prove the proposition by induction on g. When $g = 0$, there is necessarily a component for which $s = 1$ and so we give the value 0 to the unique bounded edge. We do this for all the vertices with $s = 1$. Then remove all the vertices with $s = 1$ and the unbounded edges emanating from them. Now we have another tree and so again there are vertices of $s = 1$. Two of the edges emanating from any of them have the value 0, and so the value of the last edge must be 0 by the rule. By induction, we have $\dim H^0(C_{k-1}, \omega_{C_{k-1}}) = 0$ in this case.

Now assume that we proved $\dim H^0(C_{k-1}, \omega_{C_{k-1}}) = g$ for $g \leq g_0 - 1$, with $g_0 \geq 1$. Consider a tropical curve (Γ, h) of genus g_0. Since h is an embedding by Assumption B, we identify Γ and the image $h(\Gamma)$.

Let E be an edge which is a part of the loops of Γ. Cutting E at the middle and extending both ends to infinity, we obtain a curve Γ' with genus $g_0 - 1$.

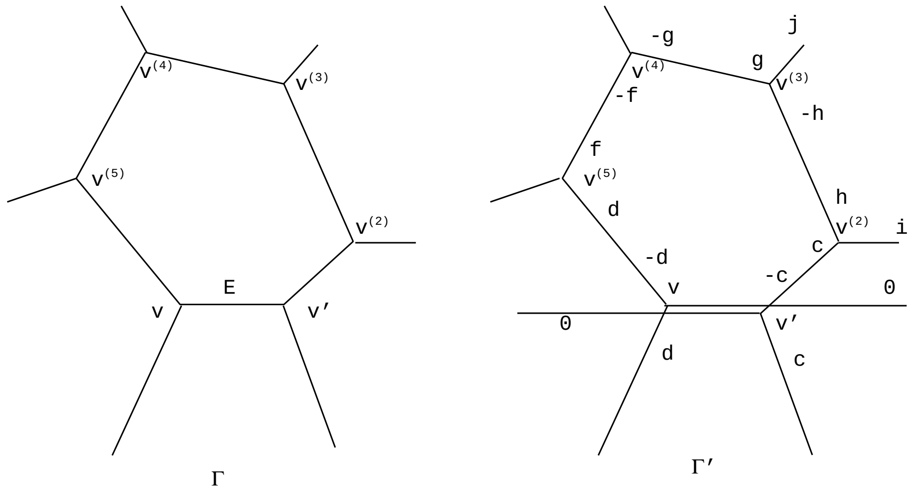

Figure 1.

By induction hypothesis, there is $g_0 - 1$ dimensional freedom to give numbers to the flags of Γ' compatibly. Let v, v' be the vertices of E and choose one of the cycles of Γ which contains E. Because v, v' has at most $s = 2$ in Γ', the numbering around v, v' looks like Figure 1. Here $c, d, f, g, h, i, j \in \mathbb{C}$, and when $s = 1$ at v or v', then d or c must be 0, respectively.

Now let us return to Γ. First let us give a value 0 to the flags $(v, \overline{vv'})$ and $(v', \overline{v'v})$ and give the same values $\pm c, \pm d$, *etc.* as Γ' to the remaining flags of Γ. This is a compatible numbering of Γ. Then give an arbitrary value b to the flag $(v, \overline{vv'})$, and add values $-b, b, -b, \cdots$ successively to the adjacent flags of the cycle.

These again give compatible numberings of Γ, which have one more freedom given by the the value of b, compared to the numberings of Γ'. So we have

$$\dim H^0(C_{k-1}, \omega_{C_{k-1}}) \geq g_0.$$

Conversely, assume $\dim H^0(C_{k-1}, \omega_{C_{k-1}}) \geq g_0 + 1$. Let $\{f_k\}$ be the set of flags of Γ and $S = \sum \mathbb{R}\langle f_k \rangle$ be the linear space of real functions on this set. We write elements of S by $\sum a_k \langle f_k \rangle$, $a_k \in \mathbb{R}$. Note that the space T of compatible numberings is a linear subspace of S. By assumption, this subspace has dimension not less than $g_0 + 1$. Choose any flag $f = f_0 = (v_0, E_0)$ which is a part of some cycle of Γ. The

hyperplane $a_0 = 0$ cuts T so that the intersection U is a linear subspace of dimension not less than g_0. Now cut the edge E_0 at the middle point, and extend both of the ends to infinity (as in Figure 1). Then we obtain a trivalent graph Γ' of genus $g_0 - 1$. It is easy to see that each element of U gives a compatible numbering of Γ'. On the other hand, by induction hypothesis, the dimension of the space of compatible numberings of Γ' is equal to $g_0 - 1$. This is a contradiction. So we have $\dim H^0(C_{k-1}, \omega_{C_{k-1}}) \leq g_0$. This proves the proposition. $\square$

Proof of Theorem 1. Let the dimension of $H^0(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})$ be d_1 and the dimension of $H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})$ be d_2. As in [9], d_1 is the same as the dimension of the moduli space of the corresponding tropical curve. By the long exact sequence, we have

$$d_1 - d_2 = n + (3g - 3 + e) - ng,$$

which is just the expected dimension of the moduli space of the corresponding tropical curve. So the tropical curve is non-superabundant if and only if $d_2 = 0$. In this case, the obstruction $H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})$ is cancelled by the freedom of the moduli of the curve $H^1(C_{k-1}, \Theta_{C_{k-1}/O_{k-1}})$. So we can lift φ_{k-1} to φ_k. Having the existence of such a lift, we can apply the proof of [9] verbatim to show that the corresponding prelog curves actually deform into smooth curves, and that the tropical curves and corresponding holomorphic curves have the same dimensional moduli space. This proves Theorem 1. $\square$

4. Combinatorial description of the dual space of obstructions

Applying the same type of combinatorics introduced in the proof of Proposition 25, we can analyze $H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})$. As we saw, if $H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})$ vanishes, we know that the prelog curves corresponding to the tropical curve can be smoothed. In this section, we give an effective method to calculate $H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})$ when it does not vanish (Theorem 30).

Remark 26. *In this section, we again perform the calculation assuming (Γ, h) is immersive for notational simplicity. However, the result in this section is straightforwardly extended to the case when (Γ, h) satisfies Assumption A, because essentially only a neighbourhood of the loops of Γ affects the calculation, and Assumption A assures that h is immersive around the loops.*

In particular, we identify the graph Γ and its image $h(\Gamma)$.

By Serre duality, we have

$$H^1(C_{k-1}, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}}) \cong H^0(C_{k-1}, (\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_C)^\vee.$$

Recall $\Theta_{C_{k-1}/O_{k-1}} \cong \mathcal{O}(-1)$ and $\omega_C \cong \mathcal{O}(-2+s)$ on each irreducible component of C_{k-1}, here s is the number of nodes of the component. From this, it is easy to see that when $s=1$,

$$\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_C) = 0$$

when restricted to that component.

When $s=2$, we have

$$\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_C) \cong \Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee)$$

on the corresponding component. Note the following inclusion:

$$(\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \subset (\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})^\vee \cong N_{\mathbb{C}}^\vee \otimes \mathcal{O}_{C_{k-1}}.$$

Let ℓ be a component of C_{k-1} and let v be the vertex of the tropical curve corresponding to ℓ. The edges emanating from v span the two dimensional subspace V_v of $N_{\mathbb{C}}$. Then it is clear that $\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee)$ is given by the subspace $V_v^\perp \subset N_{\mathbb{C}}^\vee$ tensored by $\mathbb{C}[t]/t^k$. Namely, under the convention as in the proof of Proposition 25, it is given by:

(a) Give 0 to the flag (v, E_0), where E_0 is the unique unbounded edge emanating from v.
(b) Give $\pm\alpha$, where $\alpha \in V_v^\perp$, to the remaining flags associated to v.

Let us consider the case $s=3$. For simplicity, let us first assume $n=2$. Let ℓ and v as above. In this case,

$$(\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}_{k-1}}/\Theta_{C_{k-1}})^\vee \cong \mathcal{O}(1)^\vee \cong \mathcal{O}(-1).$$

So $\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}})$ is isomorphic to $\mathbb{C}[t]/t^k$ on ℓ. On the other hand,

$$\begin{aligned}\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}}) &\subset \Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}})^\vee \otimes \omega_{C_{k-1}}) \\ &\cong N_{\mathbb{C}}^\vee \otimes (\mathbb{C}[t]/t^k)\langle \sigma, \tau\rangle\end{aligned}$$

on this component. Here σ, τ are base vectors of the space of holomorphic 1-forms on $\mathbb{P}^1$ allowing logarithmic poles at three marked points, which we used in the proof of Proposition 25.

Let

$$(la, lb), \quad (mc, md), \quad (-la-mc, -lb-md)$$

be the slopes of the edges of the tropical curve Γ emanating from v. Here $l, m \in \mathbb{Z}_{>0}$ are weights and $(a,b), (c,d)$ are primitive integral vectors. Recall that these edges correspond to the intersections of the line in the toric surface (defined by the two dimensional fan given by the tropical curve with one vertex v) with the toric divisors (see [9, Definition 5.1]). We set an inhomogeneous coordinate z on the line so that $(la, lb), (mc, md), (-la-mc, -lb-md)$ correspond to 0, 1 and ∞, respectively. As in the proof of Proposition 25, we can take σ, τ and local frames of the sheaf at the marked points so that $\sigma(0) = -\sigma(\infty) = 1, \sigma(1) = 0$ and $\tau(0) = 0, \tau(1) = -\tau(\infty) = 1$.

Lemma 27. *Let u_1, u_2 be the generators of $(\mathbb{R}\cdot(a,b))^\perp$, $(\mathbb{R}\cdot(c,d))^\perp$ in $N_\mathbb{R}^\vee$ such that $u_1((c,d)) = u_2((a,b)) = 1$. Then the space of sections $\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}})$ is given by*

$$\mathbb{C}[t]/t^k \cdot \langle lu_1\sigma - mu_2\tau\rangle \subset N_\mathbb{C}^\vee \otimes (\mathbb{C}[t]/t^k)\langle \sigma, \tau\rangle.$$

Proof. The stalks of $\Theta_{C_{k-1}/O_{k-1}}$ at 0, 1, ∞ are spanned by $(la, lb), (mc, md)$, and $(-la - mc, -lb - md)$, respectively, considered as subsets of $N_\mathbb{C} \otimes \mathcal{O}_{C_{k-1}}$. Sections of $H^0(C_{k-1}, (\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}})$ must annihilate these, and this condition determines the mentioned subspace in the statement. □

From this lemma, when we represent the sections in $\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}})$ by a trivalent vertex with some values given to the flags as in the proof of Proposition 25, we are forced to give the values $flu_1, -fmu_2$ and $f(-lu_1 + mu_2)$ of $N_\mathbb{C}^\vee \otimes (\mathbb{C}[t]/t^k)$ to the edges corresponding to $0, 1$ and ∞, respectively. Here f is an element of $(\mathbb{C}[t]/t^k)$.

From this, one sees that in the general case where n is not necessarily two, $\Gamma((\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}})$ is described as follows.

Namely, fix an inhomogeneous coordinate z on the rational curve ℓ with three marked points $0, 1, \infty$, and take sections σ, τ of $\mathcal{O}(1)$ as above. Let E_1, E_2 and E_3 be the three edges of Γ emanating from the corresponding vertex v of Γ and let w_1, w_2, w_3 be their weights. Let $n_1, n_2, n_3 \in N$ be the primitive integral generators of these edges, and let $V_1, V_2, V_3 \subset (N_\mathbb{C})^\vee$ be the subspaces which are the annihilators of $\mathbb{C}\cdot n_1$, $\mathbb{C}\cdot n_2$ and $\mathbb{C}\cdot n_3$, respectively. Then, one sees the following.

Lemma 28. *The space $H^0(\ell, (\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}})$ is naturally identified with the subspace*

$$\{(\mathbb{C}[t]/t^k)\langle w_1v_1\sigma - w_2v_2\tau\rangle | v_1 \in V_1, v_2 \in V_2, v_1(n_2) = v_2(n_1) = 1\}. \tag{3}$$

of $(N_\mathbb{C})^\vee \otimes \mathbb{C}[t]/t^k\langle\sigma, \tau\rangle$. □

Note that elements in

$$\{(\mathbb{C}[t]/t^k)\langle w_1v_1\sigma - w_2v_2\tau\rangle | v_1 \in V_1, v_2 \in V_2, v_1(n_2) = v_2(n_2) = 1\}$$

automatically annihilates $\mathbb{R}\cdot n_3$ at $z = \infty$, noting $w_3n_3 = -w_1n_1 - w_2n_2$. Also note that the sum of the values of a section of $H^0(\ell, (\varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_{k-1}/O_{k-1}})^\vee \otimes \omega_{C_{k-1}})$ at $0, 1, \infty$ is $0 \in (N_\mathbb{C})^\vee \otimes \mathbb{C}[t]/t^k$.

Using these results, we can combinatorially describe the space of obstructions. As in the proof of Proposition 25, we give values to the flags of Γ and impose compatibility conditions to the bounded edges. But this time, the value is in $N_\mathbb{C}^\vee \otimes \mathbb{C}[t]/t^k$, not just a complex number. Let $L = \cup_i L_i$ be the loops of Γ (Definition 12), where L_i are connected components. This is a closed subgraph of Γ. Let $\Gamma_T = \Gamma \setminus L$. A connected component of Γ_T is a tree. There are two types of these connected components, namely:

(U) The component contains only one flag whose vertex is contained in a loop.
(B) Otherwise.

By inductive argument, it is easy to see that all the flags in a component of type (U) must have the value zero, including the unique flag whose vertex is contained in a loop. For the type (B) too, we have the following result.

Lemma 29. *All the flags of a component of type* (B)*, including the flags whose vertices are contained in the loops, must have the value* $0 \in N_{\mathbb{C}}^{\vee} \otimes \mathbb{C}[t]/t^k$*.*

Proof. Note that Γ can be written in the following form (Figure 2).

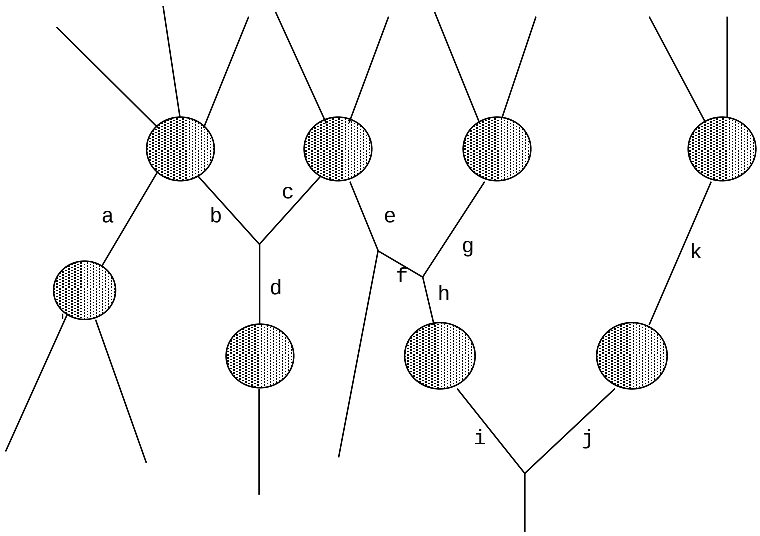

FIGURE 2.

Here, each colored disk corresponds to some component L_i of the loops of Γ. By definition of $\{L_i\}$, if we regard these disks as vertices, we obtain another tree Γ'.

Recall the above remark that all the edges contained in the components of type (U) have the value zero. In the figure above, this means that all the edges (outside the colored disks) except the ones labeled by $a, b, c, \dots, k$ have the value zero. We call the edges outside the colored disks as the *bridges*.

Now, by the fact that Γ' is a tree, it is easy to see that there is a colored disk such that the bridges emanating from it have the value zero except one bridge. Let us call this bridge r and call the remaining bridges as $a_1, \dots, a_s$. By the condition that the sum of the values of the three edges emanating from each vertex (of original Γ) is zero, we see that the sum of the values attached to $a_1, \dots, a_s, r$ is zero. Since $a_1, \dots, a_s$ have value zero, it follows that r has also the value zero. By

induction on the number of colored disks, we see that all the bridges, and so all the edges of the components of type (B) also have the value zero. $\square$

According to this lemma, we only need to consider the flags contained in some bouquet (i.e., a connected component of L). Let L_i be a bouquet. This is a graph with bivalent and trivalent vertices. Every trivalent vertex v of L_i determines a two dimensional subspace of $N_{\mathbb{C}}$ spanned by the edges emanating from it. We write it by V_v as before. Also, every edge E of L_i determines a one dimensional subspace of $N_{\mathbb{R}}$.

Let us describe the space $H = H^1(C_{k-1}, \phi_{k-1}^* \Theta_{\mathfrak{X}/\mathbb{C}} / \Theta_{C_{k-1}/O_{k-1}})^{\vee}$. Let $\{v_i\}$ be the set of trivalent vertices of L. Cutting L at each v_i, we obtain a set of piecewise linear segments $\{l_m\}$. Let U_m be the linear subspace of $N_{\mathbb{R}}$ spanned by the direction vectors of the segments of l_m. The following theorem follows from the argument so far. As we noted in Remark 26, we state this for tropical curves satisfying Assumption A, which are not necessarily immersive.

Theorem 30. *Let (Γ, h) be a tropical curve satisfying Assumption A. Elements of the space H are described by the following procedure.*

- *(I) Give the value zero to all the flags not contained in L.*
- *(II) Give a value u_m in $(U_m)^{\perp} \otimes \mathbb{C}[t]/t^k \subset (N_{\mathbb{C}})^{\vee} \otimes \mathbb{C}[t]/t^k$ to each of the flags associated to the edges of l_m.*
- *(III) The data $\{u_m\}$ give an element of H if and only if the following conditions are satisfied.*
  - *(a) At each vertex v of Γ,*
  $$u_1 + u_2 + u_3 = 0$$
  holds as an element of $(N_{\mathbb{C}})^{\vee} \otimes \mathbb{C}[t]/t^k$. Here u_1, u_2, u_3 are the data attached to the three flags in Γ which have v as the vertex.
  - *(b) The data $\{u_m\}$ is compatible on each edge of l_m, in the sense that the sum of the values attached to the two flags of an edge of l_m is zero.* $\square$

Remark 31. *We see that it almost suffices to check the conditions only at the trivalent vertices of L. The conditions (I) and (III) (a) implies that at a divalent vertex of L, the values u, u' associated to the relevant two flags satisfy $u + u' = 0$. Together with the condition (III) (b), we see that on each l_m, the values associated to the flags are unique up to sign.*

The following is immediate from this, because when the genus of Γ is one, there is no trivalent vertex in L.

Corollary 32. *When Γ is a tropical curve of genus one, then $H \cong U^{\perp} \otimes \mathbb{C}[t]/t^k$, here U is the linear subspace of $N_{\mathbb{R}}$ spanned by the direction vectors of the segments of the cycle of Γ.* $\square$

Remark 33. *(i) If we do not assume Assumption A, more degenerate situations appear. For example:*
- *A loop of Γ is mapped to a tree.*
- *Some edges are contracted to a vertex in a loop. So a vertex of a loop of $h(\Gamma)$ has higher valency.*

If h is not too much degenerate (e.g., contracts a loop to a vertex), one can extend Theorem 30 to these cases by modifying the definition of U_m above.

(ii) These degenerate cases occur in higher codimension in the subspace of the space of smoothable tropical curves (Definition 23), see Lemma 48. So these are irrelevant to the (generic) enumeration of genus one curves Theorem 71.

(iii) For higher genus cases, there are a few problems which do not appear in the genus one case.
- *More degenerate case, including the case where some loops are contracted to a vertex may appear (see Example 50).*
- *There are cases that the space of smoothings of the pre-log curves corresponding to the tropical curve has strictly larger dimension than expected in an essential way (i.e., not by the reason that the tropical curve is contained in some subspace of $N_{\mathbb{R}}$. See Examples 79, 82). In these cases, although there is locally a correspondence of the moduli spaces between tropical and holomorphic curves, the relation of the counting numbers to Gromov-Witten type invariants is unclear. Also, it is not clear whether these counting numbers are invariant under the change of the incidence conditions.*

4.1. **Example.** Let us consider genus two immersive tropical curves Γ_1 and Γ_2 in $\mathbb{R}^3$ given in Figure 3.

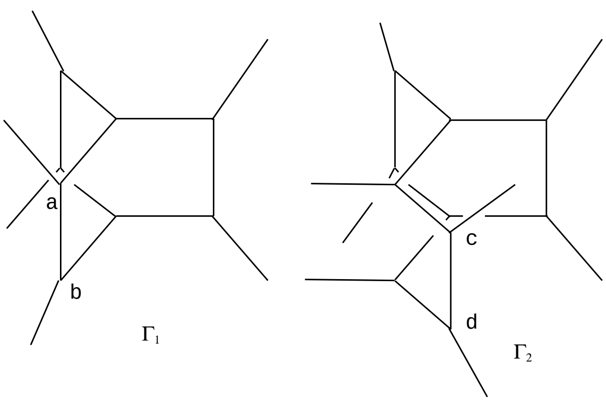

FIGURE 3.

The curve Γ_1 has six unbounded edges of directions

$$(1,0,1),\ (1,0,-1),\ (-1,-1,1),\ (-1,-1,-1),\ (0,-1,1),\ (0,-1,-1).$$

The bounded edges are:

- Three parallel vertical edges of direction $(0,0,1)$.
- Three pairs of parallel edges of directions
$$(1,0,0),\ \ (-1,-1,0),\ \ (0,1,0),$$
respectively.

The curve Γ_2 is a modification of Γ_1 at the vertices a and b. Namely:

(1) Delete the edge $\overline{ab}$ (as well as the neighboring unbounded edges).
(2) Add a pair of parallel unbounded edges of direction $(-1,0,0)$, and a pair of parallel bounded edges of direction $(1,-1,0)$ of the same length.
(3) Connect the end points c,d of the bounded edges added in (2) by a segment of direction $(0,0,1)$.
(4) Add unbounded edges of direction $(1,-1,1)$, $(1,-1,-1)$ at the vertices c,d, respectively.

Using Theorem 30, it is easy to see that Γ_1 is superabundant, while Γ_2 is non-superabundant. Namely, the set of piecewise linear segments $\{l_m\}$ of these tropical curves are given by the following three components, respectively (Figure 4).

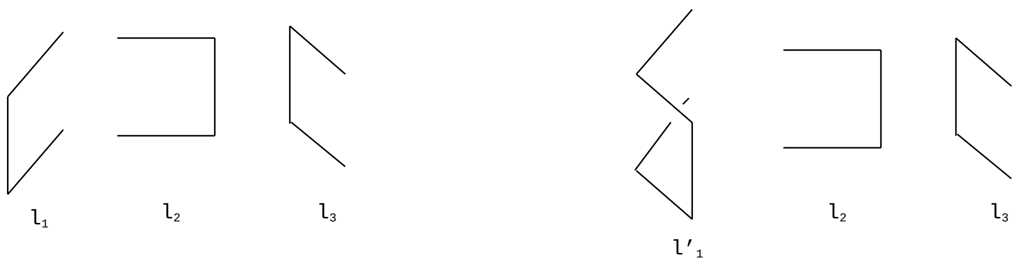

FIGURE 4.

We write the corresponding linear subspaces of $N_{\mathbb{R}} \cong \mathbb{R}^3$ by U_{l_1}, U_{l_2}, etc.. Then, using standard metric on $\mathbb{R}^3$ to identify it with its dual,
$$(U_{l_1})^{\perp} \cong \mathbb{R}\cdot(1,0,0),\ \ (U_{l_2})^{\perp} \cong \mathbb{R}\cdot(0,1,0),\ \ (U_{l_3})^{\perp} \cong \mathbb{R}\cdot(1,1,0).$$
Then it is easy to see that the space H for Γ_1 is a one dimensional vector space. Thus, Γ_1 is superabundant.

On the other hand, since $U_{l'_1} \cong \mathbb{R}^3$, $(U_{l'_1})^{\perp} = \{0\}$. From this, it is easy to see that the space H for Γ_2 is $\{0\}$. Therefore, Γ_2 is non-superabundant.

5. Correspondence theorem for superabundant curves I: Existence of smoothings for genus one curves

Having described the (dual) space H of obstructions, we want to find a condition under which they vanish. We know that the deformation of tropical curve is governed by $H^0(C_0, \varphi_{k-1}^*\Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_0/O_0})$, and there is

no need to consider obstruction. However, we also know that the corresponding complex curves (or pre-log curves) actually have obstructions for smoothing, and the smoothability cannot be determined just from the cohomology. We have to calculate the *Kuranishi map*, and this is what we do in the rest of the paper.

In this section and the next, we treat the case of genus one, which is easier partly because there is no trivalent vertex in the loop L, so the condition (III) of Theorem 30 is vacuous. In this case, the Kuranishi map takes a simplified form (Proposition 55).

Let (Γ, h) be a trivalent genus one superabundant tropical curve in $N_{\mathbb{R}} \cong \mathbb{R}^n$. *We assume that the direction vectors of the edges of* Γ *span* $\mathbb{R}^n$ (otherwise, take an affine subspace of $N_{\mathbb{R}}$ so that this condition is satisfied). Let $\varphi_0 : C_0 \to X_0$ be a generic pre-log curve corresponding to (Γ, h). We assume (Γ, h) is defined over $\mathbb{Z}$. Let L be the loop of Γ. Let A be the minimal dimensional affine plane of $\mathbb{R}^n$ which contains $h(L)$, and let $\bar{A}$ be the subspace parallel to A. The subset $A \cap h(\Gamma)$ of $h(\Gamma)$ may have several connected components, and let $h(\Gamma')$ be the unique component containing $h(L)$. By Assumption A, one sees that Γ' is a connected subgraph of Γ (see Figure 5). Because (Γ, h) is superabundant, Γ' necessarily has one-valent vertices. Let $\{\alpha_i\}$ be these vertices. If we remove $\{\alpha_i\}$ from Γ' and extend the open edges to infinity, we have a tropical curve in the affine plane A which is non-superabundant, so there is no obstruction to the smoothing of the pre-log curve corresponding to it. So, the first possible obstructions appear when we try to extend the smoothing of the node corresponding to the edge of Γ' attached to some $\{\alpha_i\}$.

5.1. **The immersive case.** *In this subsection, we assume that h is immersive.* So we identify Γ and the image $h(\Gamma)$. In particular, all the vertices of $h(\Gamma)$ is trivalent and the weight of each edge of $h(\Gamma)$ is a single integer.

Now, take one vertex α from $\{\alpha_i\}$ and let E be the edge of Γ' attached to α. Note that there is a unique path from α to the loop L. Assume that the integral length of this path is the shortest among the one-valent vertices of Γ'. Let β be the other vertex of E, see Figure 5.

Let t be the pull-back to $\mathfrak{X}$ of the standard coordinate on the base space of the family $\mathfrak{X} \to \mathbb{C}$. Let $p \in C_0$ be the node corresponding to E.

First, we consider the following basic case. Namely, we assume:

$(*)$ There is a basis of N such that the edges emanating from β are spanned by the vectors

$$(1, 0, 0, 0, \dots, 0), \quad (0, 1, 0, 0, \dots, 0), \quad (-1, -1, 0, 0, \dots, 0),$$

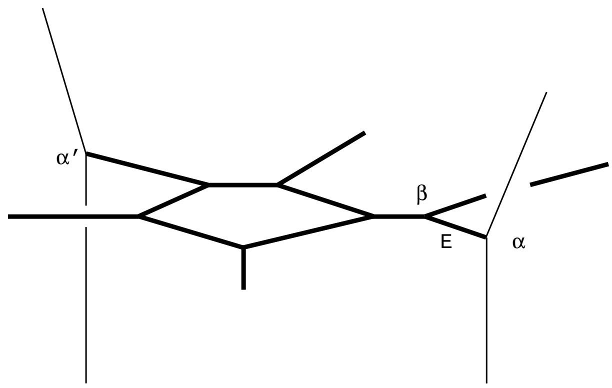

FIGURE 5. The part drawn by bold lines is $h(\Gamma')$

where the edge E is spanned by $(1, 0, 0, 0, \dots, 0)$. Similarly, the edges emanating from α are spanned by the vectors

$$(-1, 0, 0, 0, \dots, 0), \ \ (0, 0, 1, 0, \dots, 0), \ \ (1, 0, -1, 0, \dots, 0).$$

We also assume that all the edge weights are one.

In this case, locally around the node p, we can take a coordinate system $\{x, y, z, w_1, \dots, w_{n-2}\}$ of $\mathfrak{X}$ around $\varphi_0(p)$ and coordinates S, T of the branches of C_0 around p with the following properties.

(i) Each of $\{x, y, z, w_1, \dots, w_{n-2}\}$ is a character of the big torus action on $\mathfrak{X}$.

(ii) The equation $xy = t^r$ holds, here r is the integral length of the edge E. Since X_0 is given by $\{t = 0\}$, the equations $x = 0$, $y = 0$ determine the irreducible components of X_0 around $\varphi_0(p)$. Let us write the reduced structure of the variety $\{y = 0\} \cap X_0$ by $X_{0,\alpha}$, which corresponds to the vertex α. Similarly, write $X_{0,\beta}$ for the component of X_0 corresponding to the vertex β.

(iii) $\varphi_0^*(x) = S$, $\varphi_0^*(y) = T$.

(iv) Let ℓ_α be the component of C_0 corresponding to α. This is mapped to $X_{0,\alpha}$ by φ_0. The defining equations of the image are given by

$$kx + lz + m = 0, \ \ y = 0, \ \ w_1 = -\frac{\mu}{\lambda}, \ \ w_2 = a_2, \dots, w_{n-2} = a_{n-2}.$$

Here k, l, m, a_i are generic complex numbers (in particular, nonzero).

(v) Similarly, if ℓ_β is the component of C_0 corresponding to the vertex β, the defining equations of the image are given by

$$\kappa y + \lambda w_1 + \mu = 0, \ \ x = 0, \ \ z = -\frac{m}{l}, \ \ w_2 = a_2, \dots, w_{n-2} = a_{n-2}.$$

Here κ, λ, μ are generic complex numbers.

Around $\varphi_0(p)$, the log tangent bundle of $\mathfrak{X}$ is spanned by

$$x\partial_x, \ \ y\partial_y, \ \ z\partial_z, \ \ w_1\partial_{w_1}, \dots, w_{n-2}\partial_{w_{n-2}}.$$

On the other hand, we have the identity

$$ydx + xdy = rt^{r-1}dt.$$

In particular, on X_0, $ydx + xdy = 0$. Moreover, on the image $\varphi_0(\ell_\alpha)$, we have identities

$$kdx + ldz = 0, \quad dy = dw_1 = \cdots = dw_{n-2} = 0.$$

Similarly, on the image $\varphi_0(\ell_\beta)$,

$$\kappa dy + \lambda dw_1 = 0, \quad dx = dz = dw_2 = \cdots = dw_{n-2} = 0.$$

From these, we see that, around $\varphi_0(p)$, the fibers of the tangent bundle of $\varphi_0(\ell_\alpha)$ are spanned by the vector

$$x\partial_x - y\partial_y - \frac{kx}{lz} \cdot z\partial_z$$

as a subbundle of $\Theta_{\mathfrak{X}}$. Similarly, the tangent bundle of $\varphi_0(\ell_\alpha \cup \ell_\beta)$ is spanned by

$$x\partial_x - y\partial_y - \frac{kx}{lz} \cdot z\partial_z + \frac{\kappa y}{\lambda w_1} \cdot w_1\partial_{w_1}.$$

We use these calculations to construct lifts of the curve

$$\varphi_0 : C_0 \to X_0$$

to curves in $X_t, t \neq 0$, order by order with respect to t. Before doing so, we study what changes are needed when the assumption $(*)$ at the beginning of this subsection is removed.

5.1.1. *The cases of general edge directions and weights.* As long as (Γ, h) is immersive, the neighborhood of the edge E is obtained as a result of an integral affine transformation of the basic case considered in the assumption $(*)$.

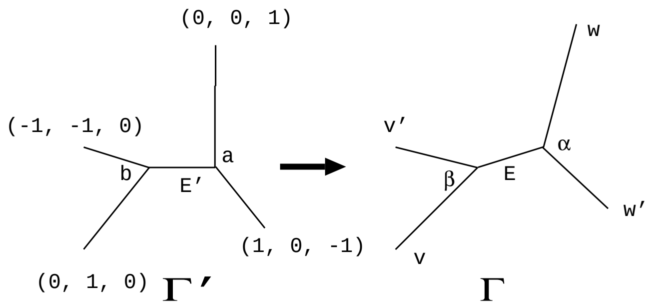

Figure 6.

For simplicity, we assume $N \cong \mathbb{Z}^3$. In Figure 6, an integral affine transformation $L : N \to N$ is uniquely determined by the following properties:

- L is linear when a and α are regarded as origins. Let $\overline{L}$ be the linear transformation obtained in this way.

- $\overline{L}$ transforms the vector $(0,1,0)$ to v, $(0,0,1)$ to w and $(-1,0,0)$ to the vector u spanning the edge E.

Note that the vectors u, v and w (as well as v', w') may not be primitive. They are multiples of primitive vectors by their edge weights. The integral length of the edge E is also some integer multiple of the integral length of the edge E'.

Remark 34. *As in* [9]*, when the weight of the edge is larger than one, we magnify h by some positive integer so that the integral length of the edge becomes an integer multiple of w.*

Let

$$f_1, f_2, f_3$$

be the dual basis in N^* of the basis $(1,0,0),(0,1,0),(0,0,1)$ in N. The tropical curve Γ' corresponds to a line in $X' = \mathbb{P}^3$. Consider a toric degeneration $\mathfrak{X}' \to \mathbb{C}$ of $\mathbb{P}^3$ defined respecting Γ'. As we argued above, there are components $X'_{0,a}, X'_{0,b}$ of X'_0, corresponding to the vertices a, b. Moreover, there are coordinate systems $\{x, w, z\}$ of $X'_{0,a}$ and $\{y, w, z\}$ of $X'_{0,b}$ corresponding to the vectors f_1, f_2, f_3, by which a generic pre-log curve corresponding to Γ' can be written in the above form (properties (iv) and (v) of the previous subsection).

On the other hand, let

$$g_1, g_2, g_3$$

be the dual basis in $N^*_{\mathbb{Q}}$ of the vectors u, v, w in N. One sees that the multiple of these vectors

$$\det \overline{L} \cdot g_1, \ \det \overline{L} \cdot g_2, \ \det \overline{L} \cdot g_3$$

belong to N^*.

The tropical curve Γ corresponds to a curve in a toric variety $X = \mathbb{P}$ given as a transformation of $\mathbb{P}^3$ by the map Φ_L induced by $\overline{L}$. Consider a toric degeneration $\mathfrak{X} \to \mathbb{C}$ of $\mathbb{P}$ defined respecting Γ. As in the case of $\mathfrak{X}'$, there are components $X_{0,\alpha}, X_{0,\beta}$ of the central fiber X'_0 corresponding to the vertices α, β of Γ. There are functions $\{X, W, Z\}$ on $X_{0,\alpha}$ and $\{Y, W, Z\}$ on $X_{0,\beta}$ corresponding to the vectors $\det \overline{L} \cdot g_1$, $\det \overline{L} \cdot g_2$, $\det \overline{L} \cdot g_3$. Note that these functions may not be (parts of) coordinate systems, when $\det \overline{L} \neq \pm 1$.

In fact, a generic pre-log curve corresponding to Γ is given by a reduced structure on some component of the curve given by an equation in the form

$$\overline{k}X + \overline{l}Z + \overline{m}_1 U_1 + \cdots + \overline{m}_c U_c + \overline{m}_{c+1} = 0, \ \ W = \epsilon_1,$$

on $X_{0,\alpha}$, where $\overline{k}, \overline{l}, \overline{m}_1, \dots, \overline{m}_{c+1}, \epsilon_1$ are generic complex numbers, and $U_1, \dots, U_c$ are functions on $X_{0,\alpha}$ which are characters of the torus action (we do not need precise form of these equations). Similarly, on $X_{0,\beta}$, a generic pre-log curve is a component of the curve written in the form

$$\overline{\kappa}Y + \overline{\lambda}W + \overline{\mu}_1 V_1 + \cdots + \overline{\mu}_d V_d + \overline{\mu}_{d+1}, \ \ Z = \epsilon_2.$$

As noted above, the linear map $\overline{L}$ induces a map Φ_L from $\mathbb{P}^3$ to $\mathbb{P}$, which is a branched covering map. Any generic curve of type Γ in $\mathbb{P}$ is obtained as the image of a line in $\mathbb{P}^3$ by Φ_L. In fact, there are $\det \overline{L}$ different lines in $\mathbb{P}^3$ which maps to such a curve in $\mathbb{P}$.

It is easy to see that we can take the degeneration $\mathfrak{X}$ so that this induces a map from $\mathfrak{X}'$ to $\mathfrak{X}$ over the base $\mathbb{C}$. Moreover, any generic pre-log curve of type Γ in X_0 is obtained as the images of pre-log curves of type Γ' in X_0' by the induced morphism. In other words, a pre-log curve $\varphi_0 : C_0 \to X_0$ splits via a pre-log curve $\varphi_0' : C_0 \to X_0'$, so that $\varphi_0 = \Phi_L \circ \varphi_0'$.

Let

$$kx + lz + m = 0, \quad y = 0,$$

and

$$\kappa y + \lambda w + \mu = 0, \quad x = 0$$

be equations of the image of a pre-log curve of type Γ'. As calculated above, the tangent bundle of the image is spanned by the vector

$$x\partial_x - y\partial_y - \frac{kx}{lz} \cdot z\partial_z + \frac{\kappa y}{\lambda w} \cdot w\partial_w.$$

The tangent bundle of a pre-log curve φ_0 of type Γ is spanned by the push-forward of these vectors by Φ_L.

The functions X, Y, Z, W are pulled back by Φ_L as follows:

$$\Phi_L^*(X) = x^{\det \overline{L}}, \ \ \Phi_L^*(Y) = y^{\det \overline{L}}, \ \ \Phi_L^*(Z) = z^{\det \overline{L}}, \ \ \Phi_L^*(W) = w^{\det \overline{L}}.$$

Then, the push-forwards of the vectors $x\partial_x, y\partial_y, z\partial_z, w\partial_w$ are:

$$(\Phi_L)_* x\partial_x = \det \overline{L} \cdot X\partial_X, \ \ (\Phi_L)_* y\partial_y = \det \overline{L} \cdot Y\partial_Y,$$
$$(\Phi_L)_* z\partial_z = \det \overline{L} \cdot Z\partial_Z, \ \ (\Phi_L)_* w\partial_w = \det \overline{L} \cdot W\partial_W.$$

So the tangent bundle of the image of φ_0 is spanned by

$$X\partial_X - Y\partial_Y - \frac{kx}{lz} \cdot Z\partial_Z + \frac{\kappa y}{\lambda w} \cdot W\partial_W.$$

Note that here we consider x, y, z, w, X, Y, Z, W as functions on C_0 by the pull-back. So this expression makes sense as a section of the pull-back of $\Theta_{\mathfrak{X}}$ to C_0.

Step 1. The zeroth order lift. By the zeroth order lift, we mean the sections of the log-normal sheaf $\mathcal{N}_{C_0/\mathfrak{X}} \cong \varphi_0^*(\Theta_{\mathfrak{X}})/\Theta_{C_0}$. Since we consider smoothings over a base space, the sections should be evaluated to one by the covector $\frac{dt}{t}$. In the following, we use the notation for the case the assumption $(*)$ is satisfied, but this step requires no change for general cases, just replacing $x\partial_x$ by $X\partial_X$, etc..

Using the basis

$$x\partial_x, \ \ y\partial_y, \ \ z\partial_z, \ \ w_1\partial_{w_1}, \dots, w_{n-2}\partial_{w_{n-2}}.$$

of $\Theta_{\mathfrak{X}}$ above and the relation

$$\frac{dx}{x}+\frac{dy}{y}=\frac{rdt}{t},$$

such sections can be represented on ℓ_α by

$$\mathfrak{n}=rx\partial_x+c(x\partial_x-y\partial_y)+c_0z\partial_z+c_1w_1\partial_{w_1}+\cdots+c_{n-2}w_{n-2}\partial_{w_{n-2}} \\ (\mathrm{mod}\ \ x\partial_x-y\partial_y-\tfrac{kx}{lz}\cdot z\partial_z)$$

(precisely speaking, the pull-back by φ_0 of these sections). On the other hand, on ℓ_β, they are represented by

$$\mathfrak{n}'=ry\partial_y+c'(x\partial_x-y\partial_y)+c_0'z\partial_z+c_1'w_1\partial_{w_1}+\cdots+c_{n-2}'w_{n-2}\partial_{w_{n-2}} \\ (\mathrm{mod}\ \ x\partial_x-y\partial_y+\tfrac{\kappa y}{\lambda w_1}\cdot w_1\partial_{w_1}).$$

To define a section over $\ell_\alpha\cup\ell_\beta$, the coefficients must satisfy

$$c_i=c_i',\ \ i=0,1,\dots,n-2$$

(note that c and c' can be different). Any section on the whole C_0 is obtained by repeating this gluing process.

In the following, we see what happens when we try to extend these lifts to non-zero order in t. Note that the vanishing of the obstruction in the case of non-superabundant curves means that all these zeroth order lifts can be extended to smoothings of any order. Since the obstruction exist only on the loop by Theorem 30, this also implies that we just have to care what happens at the loop when we extend the lifts.

Step 2. The first order lift. Here we assume the assumption $(*)$ for simplicity. Step 2 and Step 3 for general cases are dealt in Subsection 5.1.3. We also assume that the integral length of the edge $h(e)$ is one for a while.

We extend the zeroth order lifts on $\ell_\alpha\cup\ell_\beta$, and obtain a lift of $\varphi_0|_{\ell_\alpha\cup\ell_\beta}$ to a stable map over $\mathbb{C}[t]/t^2$. Recall that around p, the fibers of the sheaf $\Theta_{C_0}|_{\ell_\alpha\cup\ell_\beta}$, as a subsheaf of $\varphi_0^*(\Theta_{\mathfrak{X}})$, is spanned by the pull-back of $x\partial_x-y\partial_y-\frac{kx}{lz}\cdot z\partial_z+\frac{\kappa y}{\lambda w_1}\cdot w_1\partial_{w_1}$. Note that around $\varphi_0(p)$, the coordinates z and w_1 are not zero, so this determines a section of $\varphi_0^*(\Theta_{\mathfrak{X}})|_{\ell_\alpha\cup\ell_\beta}$ on an appropriate open subset of $\ell_\alpha\cup\ell_\beta$.

The image of one of the lifts of $\varphi_0|_{\ell_\alpha\cup\ell_\beta}$, is given by

$$kx+lz+m=0,\ \ \kappa y+\lambda w_1+\mu=0,\ \ xy=t,\ \ w_2=a_2,\dots,w_{n-2}=a_{n-2}.$$

The tangent bundle of this is again spanned by

$$(\star)\ \ x\partial_x-y\partial_y-\frac{kx}{lz}\cdot z\partial_z+\frac{\kappa y}{\lambda w_1}\cdot w_1\partial_{w_1},$$

but this time this is defined over $\mathbb{C}[t]/t^2$. Over the ring $\mathbb{C}[t]/t$, the term $\frac{kx}{lz}\cdot z\partial_z$ is zero on the component ℓ_β. However, over $\mathbb{C}[t]/t^2$, it is $\frac{k}{l}\frac{t}{yz}\cdot z\partial_z$ using $xy=t$.

Note that by definition of Γ', the span $\bar{A}$ of the direction vectors of the edges of Γ' does not contain the direction corresponding to $z\partial_z$. This implies that the vector $z\partial_z$ extends to the part of C_0 corresponding to Γ' (we write it by $C_{\Gamma'}$), giving a trivial line bundle.

Let us consider a lift of the affine curve $\ell^*_{\alpha,\beta} = Spec\,\mathbb{C}[S,T]/(ST)$,

$$\ell^{1,*}_{\alpha,\beta} = Spec\,\mathbb{C}[S,T,t]/(ST-t,t^2).$$

This is a subset of the domain of the lift of $\varphi_0|_{\ell_\alpha\cup\ell_\beta}$ mentioned above. The vector $(\star)$, when the functions x, y are pulled back to the curve by $x \mapsto S,\ \ y \mapsto T$, belongs to the sheaf

$$\mathcal{O}_{\ell^{1,*}_{\alpha,\beta}} \otimes_{\mathbb{C}} \langle x\partial_x, y\partial_y, z\partial_z, w_1\partial_{w_1}, \dots, w_{n-2}\partial_{w_{n-2}}\rangle$$

on $\ell^{1,*}_{\alpha,\beta}$, which is the natural lift of the sheaf $(\varphi_0|_{\ell^*_{\alpha,\beta}})^*\Theta_{\mathfrak{X}}$ (precisely speaking, we have to localize by (pullbacks of) z and w_1 so that $\frac{1}{z}\cdot z\partial_z$ and $\frac{1}{w_1}\cdot w_1\partial_{w_1}$ are defined in $\mathcal{O}_{\ell^{1,*}_{\alpha,\beta}}\otimes_{\mathbb{C}}\langle x\partial_x, y\partial_y, z\partial_z, w_1\partial_{w_1}, \dots, w_{n-2}\partial_{w_{n-2}}\rangle$). So the normal bundle to the image of $\ell^{1,*}_{\alpha,\beta}$ is,

$$\mathcal{O}_{\ell^{1,*}_{\alpha,\beta}}\otimes_{\mathbb{C}}\langle x\partial_x, y\partial_y, z\partial_z, w_1\partial_{w_1}, \dots, w_{n-2}\partial_{w_{n-2}}\rangle/\mathcal{O}_{\ell^{1,*}_{\alpha,\beta}}\big((x\partial_x+\frac{\kappa}{\lambda}\frac{t}{xw_1}\cdot w_1\partial_{w_1})-(y\partial_y+\frac{k}{l}\frac{t}{yz}\cdot z\partial_z)\big).$$

Though it is easy to construct a lift of $\varphi_0|_{\ell_\alpha\cup\ell_\beta}$ directly as we have just seen, it is useful to look at it from a slightly different viewpoint. Namely, given a lift over $\mathbb{C}[t]/t^2$ of $\varphi_0|_{\ell_\alpha}$ by $\mathfrak{n} = x\partial_x$, we try to extend it to ℓ_β (the space of lifts is a torsor under the vector space of sections of the normal bundle $\mathcal{N}_{\ell_\alpha/X_0}$. By 'lift by $\mathfrak{n} = x\partial_x$', we mean that we fix a base point of the torsor which is the lift of ℓ_α obtained by simply regarding the defining equations of ℓ_α as equations over $\mathbb{C}[t]/t^2$ as above.). By the description of the tangent bundle, this is equivalent to the following:

Given a germ of a rational section $\frac{k}{l}\frac{t}{yz}\cdot z\partial_z$ of the normal bundle at the node $\ell_\alpha\cap\ell_\beta$, extend it to ℓ_β in such a way that poles exist only at the nodes of C_0.

As we remarked above, since $z\partial_z$ generates a trivial line bundle on $C_{\Gamma'}$, the problem is the same as finding extensions of a germ of a rational function. Since ℓ_β is a rational curve, clearly this is possible, so a lift of $\varphi_0|_{\ell_\alpha\cup\ell_\beta}$ always exists (compare with Step 4 below).

Remark 35. *Other lifts are given by changing the values of $c, c', c_0, \cdots, c_{n-2}$ of $\mathfrak{n}$ and $\mathfrak{n}'$. This corresponds to perturbing the coefficients $k, l, m, a_2, \dots, a_{n-2}, \kappa, \lambda, \mu$ by adding constants times t, then take the lift given by $\mathfrak{n} = x\partial_x$ as above. These changes of the coefficients are reflected to the vector $(\star)$, and it is clear that $(\star)$ is changed by terms of order t^2. This observation is important (see Corollary 42 below).*

When the length r of $h(e)$ is general (but still assuming $(*)$), we replace $\ell^{1,*}_{\alpha,\beta}$ by

$$\ell^{r,*}_{\alpha,\beta,\zeta} = Spec\mathbb{C}[S,T,t]/(ST - t^r, t^{r+1}).$$

Also, the right hand side of $(\star)$ is replaced by

$$(x\partial_x + \frac{\kappa}{\lambda}\frac{t^r}{xw_1}\cdot w_1\partial_{w_1}) - (y\partial_y + \frac{k}{l}\frac{t^r}{yz}\cdot z\partial_z).$$

So when we extend the lift of ℓ_α given by $\mathfrak{n} = rx\partial_x$, it is controlled by the sections of the normal bundle, $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$, up to $k = r$, but when we consider lifts over $\mathbb{C}[t]/t^{r+1}$, the term

$$\frac{k}{l}\frac{t^r}{yz}\cdot z\partial_z$$

appears on the component ℓ_β.

Step 3. Higher order lifts. If β is a vertex of the loop of Γ, we can skip here and go to Step 4. In general, we continue the calculation as follows.

Since $\ell_\alpha \cup \ell_\beta$ has genus zero, a lift of $\varphi_0|_{\ell_\alpha\cup\ell_\beta}$ always exists, as one of which corresponding to $\mathfrak{n} = x\partial_x$ was given explicitly above. As in Remark 35, other lifts can also be written explicitly. They are stable maps from $\ell^1_{\alpha,\beta}$, the completion of the affine curve $\ell^{1,*}_{\alpha,\beta}$ we defined above.

Let $\mathcal{P}$ be the unique path from the vertex α to the loop L of Γ. By construction, the vertex β is contained in $\mathcal{P}$, and if it is not contained in L, there is a vertex on $\mathcal{P}$ other than α, which is adjacent to β. Let us write this vertex by γ. In the next step, we have to do the same calculation as above for the curve $\ell_\alpha \cup \ell_\beta \cup \ell_\gamma$, here ℓ_γ is the component of C_0 corresponding to the vertex γ. We can do this because we can still represent the image of the curve $\ell^1_{\alpha,\beta}$ explicitly, however, as we continue this to higher orders of t, it becomes hard to represent the curves explicitly, so does the calculation of sections of the normal bundles precisely. Nevertheless, it is rather easy to calculate the leading terms (that is, the lowest order terms of t), since it is determined by the data of φ_0, and not affected by the choice of the lift (Proposition 41, Corollary 42).

Now we try to extend the lift given by $\mathfrak{n} = x\partial_x$ on $\ell_\alpha \cup \ell_\beta$ to $\ell_\alpha \cup \ell_\beta \cup \ell_\gamma$. This, up to higher order terms of t, corresponds to extending the rational section

$$y\partial_y + \frac{k}{l}\frac{t}{yz}\cdot z\partial_z$$

on ℓ_β to ℓ_γ. Recall that we took the vertex α of Γ so that the direction in $\mathbb{R}^n$ corresponding to the tangent vector $z\partial_z$ is not contained in the subspace $\bar{A}$. On the other hand, since the part Γ' is non-superabundant

in A, the lift of ℓ_β given by $y\partial_y$ extends to the whole pre-log curve corresponding to Γ', giving smoothing up to any order. So the obstruction depends only on the $\frac{k}{l}\frac{t}{yz}\cdot z\partial_z$ part.

As we noted above, the vector $z\partial_z$ naturally extends to the part $C_{\Gamma'}$ of C_0 corresponding to Γ', generating a trivial line bundle which is a subbundle of the normal bundle $\mathcal{N}_{C_{\Gamma'}/X_0}$ of $C_{\Gamma'}$ in X_0. So, it suffices to extend the pull-back

$$\frac{k}{l}\frac{t}{T\cdot -\frac{1}{l}(kS+m)} = -\frac{kt}{kt+mT} = -\frac{kt}{mT}\cdot\sum_{i=0}^{\infty}(-1)^i(\frac{kt}{mT})^i$$

of the function $\frac{k}{l}\frac{t}{yz}$ on ℓ_β (recall T is an affine coordinate on ℓ_β).

Let E_1 be the edge connecting the vertices β and γ. Let q be the intersection of ℓ_β and ℓ_γ, and q_T be the value of the coordinate T at the point q. We assume that $q_T\neq\infty$ (the case $q_T=\infty$ requires few modification, see below). In this case, the image $\varphi_0(q)$ of q lies in the toric divisor of X_0 given by $w_1=0$.

We can take a coordinate system of $\mathfrak{X}$ around $\varphi_0(q)$ with the same properties as the coordinate system $\{x,y,z,w_1,\dots,w_{n-2}\}$ around $\varphi_0(p)$. In particular, there is a coordinate u such that $w_1u=t$ and an affine coordinate U on ℓ_γ which is 0 at q and u is pulled back to U by φ_0.

Because $\varphi_0(\ell_\beta)$ satisfies $\kappa y+\lambda w_1+\mu=0$ and y is pulled back to T by φ_0,

$$\varphi_0^*(w_1) = -\frac{1}{\lambda}(\kappa T+\mu).$$

Note that $q_T=-\frac{\mu}{\kappa}$.

As before, let us consider a lift of $\ell^*_{\beta,\gamma}=Spec\,\mathbb{C}[T,U]/((T-q_T)U)$,

$$\ell^{2,*}_{\beta,\gamma}=Spec\,\mathbb{C}[T,U,t]/(-\frac{\kappa}{\lambda}(T+\frac{\mu}{\kappa})U-t,t^3).$$

Using the relation $(T+\frac{\mu}{\kappa})U=-\frac{\lambda}{\kappa}t$, the function $-\frac{kt}{kt+mT}$ extends to

$$-\frac{kt}{kt+m(-\frac{\lambda t}{\kappa U}-\frac{\mu}{\kappa})} = \frac{k\kappa t}{m\mu}\cdot\sum_{i=0}^{\infty}\left(\frac{k\kappa-\frac{m\lambda}{U}}{m\mu}t\right)^i$$

on ℓ_γ. The non-constant lowest order term of t is

$$-t^2\frac{k}{m}\cdot\frac{\kappa}{\mu}\cdot\frac{\lambda}{\mu}\cdot\frac{1}{U}$$

When $q_T=\infty$, We use the coordinates $\frac{1}{w_1}$ instead of w_1, $\frac{y}{w_1}$ instead of y (and change the coordinates on C_0 correspondingly), so that $\varphi_0(\ell_\beta)$ satisfies $\kappa\frac{y}{w_1}+\lambda+\frac{\mu}{w_1}=0$, and do the same calculation.

The same process continues until we reach to the loop L. At each step, we obtain a section with ordering by t in the following form:

$$(R(t)+\frac{\chi(t)t^M}{V}+\frac{\chi_1(t)t^{M_1}}{V^2}+\frac{\chi_2(t)t^{M_2}}{V^3}+\cdots)z\partial_z,$$

where $R(t)$ is a polynomial in t with constant coefficients, $\chi(t)$ is a polynomial in t with a non-zero constant term, V is an affine coordinate of a component of C_0 (which we write by ℓ'), and M is the integral length of the path $\mathcal{P}$ from α to the vertex $v_{\ell'}$ corresponding to ℓ' in Γ. $\chi_1(t), \chi_2(t) \dots$ are also polynomials in t; $M_1, M_2, \dots$ are integers with $M < M_1 < M_2 < \cdots$.

Remark 36. *Note that this does not precisely represent the lift of φ_0, but modulo higher order terms in t than t^M. In particular, in the precise computation, not only the coefficients of $z\partial_z$ should be modified, but some terms of direction other than $z\partial_z$ may also appear. See Step 6 below and Subsection 6.1.*

If ℓ' is not a part of the loop L of Γ, the obstruction class does not yet appear (recall that we assumed that the integral distance from α to the loop L is the shortest among the one-valent vertices of Γ'), and the smoothings of φ_0 of order t^M exist and are parametrized by $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^M$ as in the genus zero case (Lemma 7.2 of [9]) or more generally, as in the non-superabundant case.

5.1.2. **Calculation of the leading term.** For later use, we look at the constant term of $\chi(t)$ a little more closely. Let ℓ be a component of C_0 and let $\sigma \in \Gamma^{[0]}$ be the corresponding vertex. The map

$$\varphi : \ell \to X_0$$

from ℓ to the corresponding irreducible component of X_0 is required to satisfy the condition that, the image $\varphi(\ell)$ is contained in the closure of an orbit of the two dimensional subtorus of the big torus acting on the components of X_0, which is determined by the subplane of $N_{\mathbb{R}}$ spanned by the edges emanating from σ. In terms of the coordinate system $\{x, y, z, w_1, \dots, w_{n-2}\}$ of $\mathfrak{X}$ we took before, the restrictions of $x, z, w_1, \dots, w_{n-2}$ compose a coordinate system on X_0, and the two dimensional torus orbit above is defined by

$$w_1 = a_1, \dots, w_{n-2} = a_{n-2},$$

where a_i are constants. Thus, noting the torically transverse condition, such maps φ are parametrized by

$$(\mathbb{C}^*)^{n-2} \times (\mathbb{C}^*)^2.$$

These factors have natural coordinate systems, namely, a coordinate system of the $(\mathbb{C}^*)^{n-2}$ factor is given by $a_1, \dots, a_{n-2}$, and a coordinate system of the $(\mathbb{C}^*)^2$ factor for $\ell = \ell_\alpha$ is given by the homogeneous coordinates $\{\frac{k}{m}, \frac{l}{m}\}$ in the above description, for example.

More generally, by the calculation above, one observes the following. Let $\mathcal{P}$ be the unique path from the vertex $\alpha_0 = \alpha$ to the loop of Γ. Let α_N be a vertex on $\mathcal{P}$ and let $\alpha, \alpha_1, \dots, \alpha_N$ be the vertices between α and α_N. Let $q_{i,i+1}$ be the node between ℓ_{α_i} and $\ell_{\alpha_{i+1}}$ (in particular,

$q_{0,1} = p$ in the notation above). In a neighborhood of each $\varphi(q_{i,i+1})$, using coordinates with the same properties as $\{x, y, z, w_1, \dots, w_{n-2}\}$ on $\mathfrak{X}$ and S, T on C_0 around $\varphi_0(p)$ and p respectively, we can do the same calculation as above. In particular, the coordinates should satisfy the following properties.

- For each i, the image $\varphi_0(\ell_{\alpha_i})$ is defined by the equations of the form
$$k_i x_i + l_i y_i + m_i = 0, \quad z_{i,1} = c_{i,1}, \dots, z_{i,n-1} = c_{i,n-1},$$
here k_i, l_i, m_i and $c_{i,j}$ are constants.
- In this notation, the node $\varphi_0(q_{i-1,i})$ corresponds to $y_i = 0$ and $\varphi_0(q_{i,i+1})$ corresponds to $x_i = 0$.
- the coordinates x_i and y_{i+1} satisfy
$$x_i y_{i+1} = t, \quad i = 0, \dots, N-1.$$

We take the functions $\frac{k_i}{m_i}, \frac{l_i}{m_i}$ for the coordinates for the $(\mathbb{C}^*)^2$ factor of the moduli space of $\varphi_0(\ell_{\alpha_i})$. With these notations, we can state the following.

Proposition 37. *The constant term of $\chi(t)$ depends only on the $(\mathbb{C}^*)^2$ factors of the above parametrizations of the curves. Explicitly, it is given by*
$$-\frac{k_0}{m_0} \cdot \frac{l_1}{m_1} \cdot \frac{k_1}{m_1} \cdot \dots \cdot \frac{l_{N-1}}{m_{N-1}} \cdot \frac{k_{N-1}}{m_{N-1}}$$

Proof. This follows straightforwardly from the calculation above. Note that this does not depend on the choices of the coordinates at each of the nodes, so long as they satisfy the above conditions. □

5.1.3. **Calculation for general cases.** In this subsection, we study what happens in the calculations so far when we remove the assumption $(*)$.

Recall the calculation in Subsection 5.1.1. A general immersive trivalent tropical curve with two vertices (Γ, h), which is not contained in an affine plane, is represented as a result of an integral affine transformation of the standard tropical curve Γ' satisfying the assumption $(*)$. Accordingly, a pre-log curve of type (Γ, h) is given as the image by the branched covering map $\Phi_L : X_0' \to X_0$ of pre-log curves of type Γ' in X_0' (we assume $\dim X = n = 3$ for notational simplicity). Such a pre-log curve in $\varphi_0' : \ell_\alpha \cup \ell_\beta \to X_0'$ is presented by the equations
$$kx + lz + m = 0, \quad y = 0,$$
$$\kappa y + \lambda w + \mu = 0, \quad x = 0.$$
Here the curve $\ell_\alpha \cup \ell_\beta$ is rational with two components and has node at one point. The tangent vector of the pre-log curve in X_0 was calculated

in Subsection 5.1.1:

$$X\partial_X - Y\partial_Y - \frac{kx}{lz}\cdot Z\partial_Z + \frac{\kappa y}{\lambda w}\cdot W\partial_W,$$

as a section of the subbundle of (the pull-back of) $\Theta_{\mathfrak{X}}$.

The functions x, y satisfy the equation

$$xy = t^r,$$

where r is the integral length of the edge E' in Figure 6. As in the case under assumption $(*)$, the vector

$$\frac{kx}{lz}\cdot Z\partial_Z = \frac{kt^r}{lyz}\cdot Z\partial_Z$$

appears when we extend the lift of the pre-log curve of type (Γ, h) to higher orders of t.

For the pre-log curve φ_0', the pull-back of the functions y, z satisfies

$$(\varphi_0')^*(y) = T,\ \ (\varphi_0')^*(z) = -\frac{1}{l}(kS+m),$$

for appropriate coordinates S, T around the node of $\ell_\alpha \cup \ell_\beta$. Substituting these to the above vector, we have, as before,

$$\frac{k}{l}\frac{t^r}{T\cdot -\frac{1}{l}(kS+m)}\cdot Z\partial_Z = -\frac{kt^r}{kt^r+mT}\cdot Z\partial_Z = -\frac{kt^r}{mT}\cdot\sum_{i=0}^{\infty}(-1)^i(\frac{kt^r}{mT})^i\cdot Z\partial_Z$$

We note the following point. There are several different pre-log curves in X_0' which is mapped to a given pre-log curve of type (Γ, h) in X_0. In particular, when the weight of the edge E in Figure 6 is w_E, then the pre-log curves in X_0' whose image is given by

$$k\zeta^{-1}x + lz + m = 0,\ \ y = 0,$$
$$\kappa\zeta y + \lambda w + \mu = 0,\ \ x = 0$$

has the same image by Φ_L as φ_0'. Here ζ is any w_E-th root of unity. The same calculation as above produces

$$-\frac{k\zeta t^r}{mT}\cdot\sum_{i=0}^{\infty}(-1)^i(\frac{k\zeta t^r}{mT})^i\cdot Z\partial_Z,$$

which depends on ζ. However, as we calculated in Step 3, the contribution to the leading term from the edge E comes in the form of product

$$\frac{k}{m}\cdot\frac{\kappa}{\mu}.$$

Then, as shown in the above equations, when k is multiplied by ζ, then κ is multiplied by ζ^{-1}, so that the total contribution does not depend on ζ. Note that though the integral length of the edge E in Figure 6 is rw, the exponent of t is r.

This is the calculation for Step 2, and the calculation for Step 3 can be done in the same manner. As we argued above, each component of

a pre-log curve of type (Γ, h) can be pulled-back to a standard form, satisfying the assumption $(*)$. It follows from the result in Subsection 5.1.2, each component is locally parametrized by

$$(\mathbb{C}^*)^{n-2} \times (\mathbb{C}^*)^2.$$

The $(\mathbb{C}^*)^2$ component of these local parameters is given by the ratio ($\{\frac{k}{m}, \frac{l}{m}\}$ in Subsection 5.1.2) of the coefficients of the defining equations of the pull-back. There may be several different pull-backs, and the local parameters depends on the choice. However, the difference is only the multiplications of roots of unity, and one sees, as in the above calculation, these dependences are cancelled pairwise.

There is another important notice, which also appeared above. Namely, the exponent of t contributed from an edge of length r and weight w is given by $\frac{r}{w}$. According to this remark, we re-define the path length of a tropical curve.

Definition 38. Let (Γ, h) be an arbitrary tropical curve defined over integers. Assume that if $E \subset \Gamma$ is an wedge with weight w_E, then the integral length r_E of the image $h(E)$ is an integer multiple of w_E (r_E can be zero). Let

$$\mathcal{P} : h(v_1) \mapsto h(v_2) \mapsto \cdots \mapsto h(v_m)$$

be a path in $h(\Gamma)$, here v_i are vertices of Γ. Let E_i be an edge connecting v_i and v_{i+1}. Then, we define the length $\ell_{(\Gamma,h)}(\mathcal{P})$ of $\mathcal{P}$ by

$$\ell_{(\Gamma,h)}(\mathcal{P}) = \sum_{i=1}^{m-1} \frac{r_{E_i}}{w_{E_i}}.$$

Note that $\ell_{(\Gamma,h)}(\mathcal{P})$ is an integer by the assumption about r_E and w_E. As in Remark 34, we always assume this condition for (Γ, h).

Remark 39. *When the tropical curve (Γ, h) satisfies Assumption A, the length of a path $\mathcal{P} \subset h(\Gamma)$ is an invariant of $\mathcal{P}$. However, for more general cases where some of the edges of Γ can be merged in the image $h(\Gamma)$, the same path $\mathcal{P}$ may have several different values of length, according to the choice of its pre-image.*

As a result of these calculations, we have a generalization of Proposition 37. Namely, continuing the calculation of the coefficient of the section $Z\partial_Z$ until we reach to the loop, we obtain

$$(R(t) + \frac{\chi(t)t^M}{V} + \frac{\chi_1(t)t^{M_1}}{V^2} + \frac{\chi_2(t)t^{M_2}}{V^3} + \cdots)Z\partial_Z,$$

at each step as before. Then the coefficient $\chi(t)$ of the leading term and the exponent M of t is given by the following:

Proposition 40. *The integer M is equal to the length $\ell_{(\Gamma,h)}(\mathcal{P})$ of the path $\mathcal{P}$ connecting α and α_N in the sense of Definition 38. The*

coefficient $\chi(t)$ *of the leading term is given by*

$$-\left(\frac{k_0}{m_0}\right)\cdot\left(\frac{l_1}{m_1}\right)\cdot\left(\frac{k_1}{m_1}\right)\cdot\dots\cdot\left(\frac{l_{N-1}}{m_{N-1}}\right)\cdot\left(\frac{k_{N-1}}{m_{N-1}}\right).$$

Here k_i, l_i, m_i *are coefficients of the defining equations of a pull-back of the component* $\varphi_0(\ell_{\alpha_i})$. □

Step 4. Obstruction at the loop. Again, we assume the assumption $(*)$. When there are edges with higher weights, the situation is almost the same, due to Proposition 40. The differences are that the coefficients must be calculated after lifting the curves to the branched covers, and that the integral length of the path must be modified as Definition 38. We omit the details of this step for these cases.

Now let α_N be the vertex of L nearest to α (it is determined uniquely). Let ℓ_{α_N} be the component of C_0 corresponding to α_N. Let E_{α_N} be the edge attached to α_N which is the last edge in the path $\mathcal{P}$ from α to α_N. Let U_N be an affine coordinate of ℓ_{α_N} whose value at the node corresponding to E_{α_N} is zero. Then, as above, we have a section

$$(R(t)+\frac{\chi(t)t^M}{U_N}+\frac{\chi_1(t)t^{M_1}}{U_N^2}+\frac{\chi_2(t)t^{M_2}}{U_N^3}+\cdots)z\partial_z$$

on ℓ_{α_N}. Here M is the length $\ell_{(\Gamma,h)}(\mathcal{P})$ of the path $\mathcal{P}$ in the sense of Definition 38.

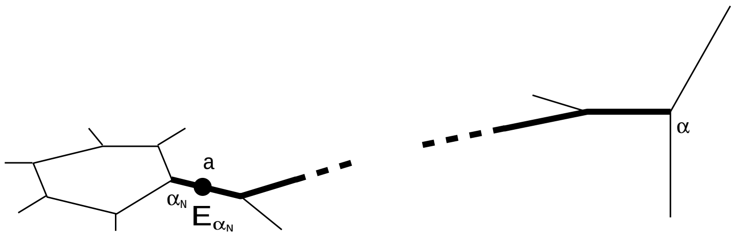

FIGURE 7. The path $\mathcal{P}$ is drawn by bold lines.

The term $\chi(t)$ was calculated in Proposition 37. However, this is not what we really want. Note that the cohomology group $H^1(C_0,\mathcal{O}_{C_0})$ is generated by a logarithmic differential form which is represented by $\frac{dU}{U}$ on each component of the loop, where U is a coordinate which takes the value 0 or ∞ at the intersection with the other components of the loop. So, we have to use such a coordinate when we calculate the residue.

As above, using appropriate coordinates, the component of $\varphi_0(\ell_{\alpha_N})$ corresponding to the vertex α_N is defined by the equations,

$$k_Nx_N+l_Ny_N+m_N=0,\ \ z_{N,1}=0,\ \ z_{N,2}=c_{N,2},\dots,z_{N,n-1}=c_{N,n-1}.$$

At this stage, we have the term $\chi(t)$ of the form

$$-\frac{k_0}{m_0}\cdot\frac{l_1}{m_1}\cdot\frac{k_1}{m_1}\cdot\dots\cdot\frac{l_{N-1}}{m_{N-1}}\cdot\frac{k_{N-1}}{m_{N-1}}\cdot\frac{t^M}{U_N}\cdot z\partial_z,$$

The function y_N is pulled back to U_N by φ_0. Then x_N is pulled back to another affine coordinate

$$U'_N = -\frac{1}{k_N}(l_N U_N + m_N).$$

Note that this coordinate satisfies the conditions about U suitable for the calculation of the residue. So when we rewrite the term $\frac{t^M}{U_N}$ by using U'_N, we have

$$\frac{t^M}{U_N} = -\frac{l_N t^M}{k_N U'_N + m_N} = -\frac{l_N t^M}{m_N} \cdot \frac{1}{1 + \frac{k_N}{m_N} U'_N}.$$

Thus, it has the first order pole at $U'_N = -\frac{m_N}{k_N}$ with residue $-\frac{l_N t^M}{m_N}$. We record this since we use it later.

Proposition 41. *The leading term of the obstruction contributed from the vertex α is given by*

$$\frac{k_0}{m_0} \cdot \frac{l_1}{m_1} \cdot \frac{k_1}{m_1} \cdot \dots \cdot \frac{l_{N-1}}{m_{N-1}} \cdot \frac{k_{N-1}}{m_{N-1}} \cdot \frac{l_N}{m_N} \cdot \frac{t^M}{1 + \frac{k_N}{m_N} U'_N} \cdot z\partial_z.$$

□

Corollary 42. *The leading term of the obstruction is determined by the configuration of the image $\varphi_0(C_0)$. Changing the lift by the sections of the normal bundle $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ changes the obstruction in higher order with respect to t.* □

When we continue to extend the section $\frac{k_0}{m_0} \cdot \frac{l_1}{m_1} \cdot \frac{k_1}{m_1} \cdot \dots \cdot \frac{l_{N-1}}{m_{N-1}} \cdot \frac{k_{N-1}}{m_{N-1}} \cdot \frac{l_N}{m_N} \cdot \frac{t^M}{1+\frac{k_N}{m_N}U'_N} \cdot z\partial_z$, there are two directions to extend, and, since L is a loop, these meet again. However, it is easy to see that these extensions do not coincide where they meet, and this is the (leading term of the) obstruction for the smoothing.

We can understand this in the following way. Namely, consider the part C_L of C_0 corresponding to the loop L. It is a nodal genus one curve (a loop composed by a chain of rational curves). Our problem of extending $\frac{k_0}{m_0} \cdot \frac{l_1}{m_1} \cdot \frac{k_1}{m_1} \cdot \dots \cdot \frac{l_{N-1}}{m_{N-1}} \cdot \frac{k_{N-1}}{m_{N-1}} \cdot \frac{l_N}{m_N} \cdot \frac{t^M}{1+\frac{k_N}{m_N}U'_N} \cdot z\partial_z$ to C_L is the same as finding a rational function on C_L which has a pole of order one whose residue is

$$\frac{k_0}{m_0} \cdot \frac{l_1}{m_1} \cdot \frac{k_1}{m_1} \cdot \dots \cdot \frac{l_{N-1}}{m_{N-1}} \cdot \frac{k_{N-1}}{m_{N-1}} \cdot \frac{l_N}{m_N} \cdot t^M.$$

Since there is no rational function on an elliptic curve which has just one pole of order one, there is no solution to this problem.

Step 5. Cancelation of the obstruction: a toy model. In Step 4, we observed that a one valent vertex of Γ' produced an obstruction for the smoothing of a pre-log curve $\varphi_0 : C_0 \to \mathfrak{X}$. We will see that the

necessary and sufficient condition for the vanishing of this obstruction can also be understood using the same idea.

The very basic model for this argument is the following. Namely, consider the exact sequence on C_L:

$$0 \to \mathcal{O}_{C_L} \to \mathcal{O}_{C_L}(\sum_{i=1}^{m} P_i) \to \oplus_{i=0}^{m} \mathbb{C}_{P_i} \to 0,$$

where $P_1, \ldots, P_m$ are divisors of C_L. Its cohomology exact sequence is,

$$0 \to H^0(C_L, \mathcal{O}_{C_L}) \to H^0(C_L, \mathcal{O}_{C_L}(\sum_{i=1}^{m} P_i)) \to \oplus_{i=0}^{m} \mathbb{C}_{P_i} \xrightarrow{s} H^1(C_L, \mathcal{O}_{C_L}) \to 0.$$

Here $H^1(C_L, \mathcal{O}_{C_L}) \cong \mathbb{C}$ and the map s is given by taking the sum. Note that the same argument is valid even if some of the P_i coincide. In this case too, the map s is given by the sum of the residues at the poles P_i. When germs of rational sections are given at $P_1, \ldots, P_m$, they can be extended to C_L if and only if their residues belong to the kernel of the map s.

Recall that we are considering the sections of the bundle generated by $z\partial_z$, which is a subbundle of the normal bundle $\mathcal{N}_{C_{\Gamma'}/X_0}$ of $C_{\Gamma'}$ in X_0. In general, we have to consider the whole normal bundle, so we replace $\mathcal{O}_{C_L}$ ($\cong \mathcal{O}_{C_L} \cdot z\partial_z$) by $\mathcal{N}_{C_{\Gamma'}/X_0}|_{C_L}$. Then we have the following exact sequence

$$\begin{aligned} 0 \to H^0(C_L, \mathcal{N}_{C_{\Gamma'}/X_0}|_{C_L}) \to \quad & H^0(C_L, \mathcal{N}_{C_{\Gamma'}/X_0}|_{C_L} \otimes \mathcal{O}_{C_L}(\textstyle\sum_{i=0}^{m} P_i)) \\ & \to \oplus_{i=0}^{m} \mathbb{C}_{P_i}^{n-1} \xrightarrow{s} H^1(C_L, \mathcal{N}_{C_{\Gamma'}/X_0}|_{C_L}) \to 0. \end{aligned}$$

The third term $\mathbb{C}_{P_i}^{n-1}$ is naturally identified with the quotient $N_{\mathbb{C}}/\mathbb{C} \cdot v_i$. Here v_i is a vector in $N_{\mathbb{C}}$ corresponding to a tangent vector of C_0 at the pole P_i, which is the node corresponding to the edge E_{α_N} (see Figure 7). Note that as a point in C_L, P_i is not a node, but a smooth (marked) point. The important property of v_i is that it is annihilated by vectors of $(\bar{A}_{\mathbb{C}})^{\perp}$, regardless of the specific position of P_i. By the calculation of the (dual space of the) obstruction in Section 4, $H^1(C_L, \mathcal{N}_{C_{\Gamma'}/X_0}|_{C_L})$ is naturally identified with $H^1(C_0, \mathcal{N}_{C_0/X_0})$ ($\cong H^1(C_0, \varphi_0^* \Theta_{\mathfrak{X}/\mathbb{C}}/\Theta_{C_0/O_0})$, in the notation in Section 4) because the obstruction only exists on the loop.

On the other hand, by Theorem 30, the dual space H of the obstruction is given by vectors in $(\bar{A}_{\mathbb{C}})^{\perp}$, so the images of s and vectors in H make natural pairings induced from the pairings between elements of $N_{\mathbb{C}}$ and $N_{\mathbb{C}}^{\vee}$. At the level of sections, this means taking the natural pairing of the direction vector $z\partial_z$ (seen as a vector in N) and the element of $(\bar{A}_{\mathbb{C}})^{\perp} \subset N_{\mathbb{C}}^{*}$, then taking the residue of the product with a generator of $H^1(C_L, \mathcal{O}_{C_L})$.

Assume first that there are only two poles and both of them have order one in the above exact sequence, and $\dim N_{\mathbb{R}} = \dim \bar{A} + 1$. Then it is easy to see, by Proposition 41 (see the proof of Theorem 45 for

details), that the existence of a local lift, whose leading term of the obstruction for extending it over C_L vanishes, is equivalent to the following: There is a one valent vertex α' of Γ', other than α, satisfying the following condition:

The integral distance from α to the loop L is the same as that from α' to L (note that we assumed that this is the shortest distance among those from the vertices of Γ' to the loop).

Step 6. Cancelation of the obstruction: Existence of smoothings. We have discussed the obstructions coming from the one valent vertices of Γ'. In general, there are obstructions coming from the vertices in $\Gamma \setminus \Gamma'$. Namely, take a connected component $\mathcal{T}$ of $\Gamma \setminus \Gamma'$. This is a tree with open ends. Let α be the one valent vertex of Γ' to which $\mathcal{T}$ is attached. Let A_1 be the minimal affine subspace of $N_{\mathbb{R}}$ containing Γ' and the edges emanating from α. If $\mathcal{T}$ is contained in A_1, then the contributions to the obstruction coming from the vertices of $\mathcal{T}$ is dominated by that from the vertex α, see the proof of Theorem 45.

So assume $\mathcal{T}$ is not contained in A_1. In this case, $\mathcal{T} \cap A_1$ may have several connected components, and let $\mathcal{T}'$ be the component closest to the loop. The set $\mathcal{T}'$ is a tree, and has several one-valent vertices. Let α' be the one valent vertex which has the minimal integral distance to the vertex α. Then do the same calculation as above to this vertex α'. Let $\bar{D}$ be the two dimensional subspace spanned by the direction vectors emanating from α', and $\bar{A}_1$ be the subspace parallel to A_1. By definition, $\bar{D}/(\bar{A}_1 \cap \bar{D})$ is one dimensional, and if $v \in \bar{D}$ is a representative of a basis of $\bar{D}/(\bar{A}_1 \cap \bar{D})$, the calculation computes the obstruction in the direction $v \bmod \bar{A}_1$.

Next, taking the minimal affine subspace A_2 containing A_1 and the edges emanating from α', continue the same process.

Summing up these obstructions contributed from one valent vertices in each of this process, one obtains the leading term of the obstruction.

On the other hand, the precise obstruction is calculated as follows, though it will be in general difficult to calculate it explicitly: Namely, Let L be the loop of Γ. The set $\Gamma \setminus L$ is a disjoint union of trivalent trees with open ends. Let $\mathcal{U}$ be one of the connected components, and e be the edge of $\mathcal{U}$ attached to the loop. Let ν be the unique vertex of L attached to e.

When the pre-log curve φ_0 is lifted to a map $\varphi_k : C_k \to \mathfrak{X}$ over $\mathbb{C}[t]/t^{k+1}$, here C_k is a lift of C_0, one calculates a generator of the tangent bundle, as a subbundle of $\Theta_{\mathfrak{X}}$, using appropriate coordinates as above. As the calculation before, around the node of C_0 corresponding to the edge e, it is written in the form

$$(x_0\partial_{x_0} - y_0\partial_{y_0}) + r,$$

here x_0, y_0 satisfies $x_0 y_0 = t^m$, the node of C_0 is mapped to a divisor of X_0 defined by $x_0 = y_0 = 0$, and the sum of the other terms r does not contain the vectors $x_0\partial_{x_0}$ nor $y_0\partial_{y_0}$. Pulling back r by φ_k and representing it using an appropriate coordinate on ℓ_ν as Step 4, one obtains a germ of a rational section, and the first order pole of it gives the obstruction. The whole obstruction is obtained by summing up the contributions from all the connected components of $\Gamma \setminus L$.

So we give the following definition. Let $\{\mathcal{U}_i\}$ be the set of connected components of $\Gamma \setminus L$. Recall that when $\varphi_k : C_k \to \mathfrak{X}$ is a lift, other lifts of the restriction of φ_k to $C_{\mathcal{U}_i}$ are parametrized by the sections of the restriction of $\mathcal{N}_{C_0/X_0} \otimes \mathbb{C}[t]/t^k$ to $C_{\mathcal{U}_i}$ (note that though every section of $\mathcal{N}_{C_0/X_0} \otimes \mathbb{C}[t]/t^k$ on C_0 need not correspond to a smoothing, however, its restriction to each $C_{\mathcal{U}_i}$ does, because this part is genus zero). When $\mathfrak{n}$ is a section of $\mathcal{N}_{C_0/X_0} \otimes \mathbb{C}[t]/t^k$ on C_0, let

$$\varphi_{k,i}(\mathfrak{n}) : C_{k,i,\mathfrak{n}} \to \mathfrak{X}$$

be the perturbation of $\varphi_k|_{C_{\mathcal{U}_i}}$ corresponding to the restriction of $\mathfrak{n}$ to $C_{\mathcal{U}_i}$. Here $C_{k,i,\mathfrak{n}}$ is an appropriate k-th order lift of $C_{\mathcal{U}_i}$.

Definition 43. We write by

$$o(\mathfrak{n}; \varphi_k)_i \in \frac{1}{U} \cdot (N_{\mathbb{C}} \otimes \mathbb{C}[t]/t^{k+1})$$

the contribution to the obstruction from the component $\mathcal{U}_i$ calculated as above, for the curve $\varphi_{k,i}(\mathfrak{n})$. Here U is, as in Step 4, a suitable affine coordinate on ℓ_ν, namely, in the notation of Proposition 41, $U = 1 + \frac{k_N}{m_N} U'_N$.

The necessary and sufficient condition for the smoothability of C_0 is the vanishing of the pairing of the obstructions (considered as germs of rational functions with data of order of t and the direction in $N_{\mathbb{R}}$) with all vectors in $(\bar{A})^\perp$. Note that each element of $(\bar{A})^\perp$ gives a hyperplane in $N_{\mathbb{R}}$, the set of vectors annihilated by that element.

From this, we can readily extend the condition at the last of Step 5 for the case of two poles to general cases. Namely, assuming (Γ, h) is immersive, and when the edge weights are one, the desired condition precisely coincides with the *well-spacedness condition* considered by Speyer [12]. When edge weights are general, it is modified using the path length of Definition 38.

Definition 44. An immersive genus one superabundant tropical curve (Γ, h) is said to be *well-spaced* if the following condition is satisfied for any affine hyperplane $\mathcal{H}$ of $N_{\mathbb{R}}$ containing $h(\Gamma')$. Let

$$\Gamma_{\mathcal{H}} = h(\Gamma) \cap \mathcal{H}$$

and let

$$p_1^{\mathcal{H}}, \dots, p_j^{\mathcal{H}}$$

be the one valent vertices of it. Denote by

$$\mathcal{P}_i, \ \ i = 1, \dots, j$$

the unique path connecting $p_i^{\mathcal{H}}$ and L. Then the set

$$\{\ell_{(\Gamma,h)}(\mathcal{P}_1), \dots, \ell_{(\Gamma,h)}(\mathcal{P}_j)\}$$

of positive integers contains at least two minimum.

Theorem 45. *Assume that* (Γ, h) *is an immersive superabundant tropical curve of genus one. Then it is smoothable (Definition 23) if and only if* (Γ, h) *is well-spaced.*

Proof. As in the main text, we assume that the direction vectors of the edges of $h(\Gamma)$ span $N_{\mathbb{R}}$. The precise obstruction explained above can be written in the form

$$o(\varphi_k) + o_1,$$

here $o(\varphi_0)$ is the sum of the leading terms, which depends only on the data of $\varphi_0(C_0)$ according to Proposition 41, and o_1 is the sum of the correction terms. Each term in $o(\varphi_0)$ has data of a direction vector in $N_{\mathbb{R}}$ and an order in t. These are determined by the tropical curve (Γ, h).

Now let us assume that the dual space of obstructions H has dimension one for simplicity. Then we can think of the obstruction as a polynomial in variable t. Suppose we perturb φ_0 by adding terms of order t to the coefficients of the defining equations of $\varphi(C_0)$. When we calculate the obstruction using these perturbed coefficients, if the terms in $o(\varphi_0)$ is modified in order t^m for some integer m, then the terms in o_1 is modified in order not less than t^{m+1}. Thus, by induction on the order of t, terms of higher order in t can be cancelled by modifying φ_0. So, for the vanishing of the obstruction, it is enough if we can choose φ_0 so that the leading term $o(\varphi_0)$ vanishes. By Proposition 41, it is easy to see that the necessary and sufficient condition for this is the well-spacedness of the tropical curve (Γ, h).

If $\dim H$ is larger than one, we apply the above argument inductively to the vertices of $\Gamma \setminus L$, starting from the ones closer to the loop, just as in the discussion at the beginning of Step 6. Then it is easy to see that, in each order of t, the well-spacedness condition is again necessary and sufficient for the existence of the configuration of a curve whose obstruction vanishes. $\square$

5.2. **General genus one case.** Here we remove the assumption that (Γ, h) is immersive. *However, we still assume Assumption A.*

In this case, there appears an important difference in the treatment of the edges with higher weights. Namely, in the previous subsection, these edges represent points of pre-log curves which intersect the toric divisors with higher multiplicities. But here, in additoin to this, the case where they represent points of pre-log curves which intersect the

toric divisors at different points appear. However, under Assumption A, we need to consider this latter possibility only for the unbounded edges. According to Remark 33 (ii) (or Lemma 48 below), this suffices for the enumeration problem.

We begin with two examples, which extend the calculation in Subsection 5.1 to the case of four-valent vertex and play important roles in the rest of this paper.

Example 1. In this example, we deal with the four valent vertex with an edge $h(E)$ whose weight $w(h(E))$ is a pair of integers (w_1, w_2). Namely, consider a tropical curve (Γ_0, h_0) given in Figure 8.

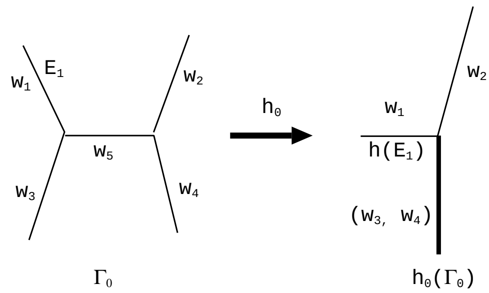

FIGURE 8. w_i are the weights of the edges.

As in Subsection 5.1, in the main text we explain the smooth case with all the weights w_i (including w_5) are one for simplicity, and remark about general cases later.

If (Γ_0, h_0) is a part of larger tropical curve, then, under Assumption A, the edge drawn by bold line (write it as $\mathfrak{E}$) of $h_0(\Gamma_0)$ must be an unbounded edge. We study what happens when we extend a local lift of the curve corresponding to $h_0(\Gamma_0)$ to the other parts, as we did in Subsection 5.1.

So as in Subsection 5.1, we consider the following situation (we use the notation in Figure 8):

- (Γ_0, h_0) is a part of a genus one superabundant tropical curve (Γ, h).
- Let L be the loop of Γ. As in Subsection 5.1, let A be the minimal dimensional affine subspace of $N_{\mathbb{R}}$ containing $h(L)$. Then the edge $h(E_1) = \mathfrak{E}_1$ is bounded, and contained in the unique connected component of $A \cap h(\Gamma)$ containing $h(L)$.
- The unbounded edge $\mathfrak{E}$ is not contained in A.

Also, we take a coordinate system $\{x, y, z, w_1, \dots, w_{n-2}\}$ on the total space $\mathfrak{X}$ of a toric degeneration defined respecting (Γ, h), satisfying the properties as in Subsection 5.1. In particular, we can assume the following properties.

Namely, let $\varphi_0 : C_0 \to X_0$ be a generic pre-log curve of type (Γ, h). Let α be the unique vertex of Γ_0 and $\varphi_0|_{\ell_\alpha} : \mathbb{P}^1 \to X_0$ be the component corresponding to α. Then $\varphi_0|_{\ell_\alpha}(\mathbb{P}^1)$ is defined by the equations

$$x^2 + ax + b + cz = 0, \quad y = 0, \quad w_1 = a_1, \dots, w_{n-2} = a_{n-2}.$$

Here $a, b, c, a_1, \dots, a_{n-2}$ are generic complex numbers, and $b, c, a_1, \dots, a_{n-2}$ are non-zero. Also, the equation $xy = t$ holds, and the node p of C_0 corresponding to the edge E_1 is mapped into the set $\{x = 0\} \cap \{y = 0\}$. In particular, $\varphi_0|_{\ell_\alpha}$ is parametrized by

$$(a; b, c; a_1, \dots, a_{n-2}) \in \mathbb{C} \times (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{n-2}.$$

By differentiating the defining equation of $\varphi_0|_{\ell_\alpha}(\mathbb{P}^1)$, we have

$$cdz + (2x + a)dx = 0,$$

and the fibers of the tangent bundle of $\varphi_0|_{\ell_\alpha}(\mathbb{P}^1)$ are spanned by

$$x\partial_x - y\partial_y - \frac{2x^2 + ax}{cz} z\partial_z \tag{4}$$

as a subbundle of $\Theta_{\mathfrak{X}}$ near $\varphi_0(p)$.

As in the calculation in Subsection 5.1, we see from this that when we extend the zeroth order lift of $\varphi_0|_{\ell_\alpha}$ given by $x\partial_x$, then the term

$$\frac{1}{cz} \cdot (2\frac{t^2}{y^2} + a\frac{t}{y}) z\partial_z = -\frac{1}{\frac{t^2}{y^2} + a\frac{t}{y} + b} \cdot (2\frac{t^2}{y^2} + a\frac{t}{y}) z\partial_z$$

appears on the neighboring component of $\varphi_0(C_0)$.

When the edge $h(E_1)$ is adjacent to the loop $h(L)$, then the first order pole $\frac{at}{by}$ vanishes when we take $a = 0$, so that the obstruction in the direction of $z\partial_z$ also vanishes. More precisely, we should use an appropriate coordinate as in Step 4 in the previous subsection, and perturb a in order t and higher so that the first order pole is cancelled.

In more general cases, as in the previous subsection, we have the terms

$$(R(t) + \frac{\chi(t)t^M}{V} + \frac{\chi_1(t)t^{M_1}}{V^2} + \frac{\chi_2(t)t^{M_2}}{V^3} + \cdots) z\partial_z,$$

in each step of extending the lift. By the discussion before Proposition 37, we observe the following. Here we use the same notation as in Proposition 37.

Proposition 46. *The constant term of $\chi(t)$ is given by*

$$\frac{a}{b} \cdot \frac{l_1}{m_1} \cdot \frac{k_1}{m_1} \cdot \dots \cdot \frac{l_{N-1}}{m_{N-1}} \cdot \frac{k_{N-1}}{m_{N-1}}.$$

The difference from the case of Proposition 37 is that we can take a to be zero, so that the leading term of the obstruction vanishes. The other terms have higher order in t, and since the constants $b, k_1, m_1, \dots$ are non-zero, we can cancel these terms by perturbing a by terms divisible by t.

Similarly, we can cancel the obstructions of direction $z\partial_z$ (mod $\bar{A}$) coming from the other vertices, provided the integral distance to the loop from these vertices are not smaller than that from the vertex α.

The general cases. Now we consider the general cases where the weights w_i are general, and the toric surface associated to the tropical curve (Γ_0, h_0) is not necessarily smooth. The idea is the same as in Subsection 5.1.3. Namely, the curves are described as the image of the standard model in the above calculation, under the branched covering map between toric surfaces and their degeneration. We describe some details.

Consider Figure 9. We take w_1, w_3, w_4 general. These numbers and the degree of (Γ_0, h_0) determine w_2 and w_5. We consider *immersed* tropical curves (Γ', h') of the same weights and degree as (Γ_0, h_0).

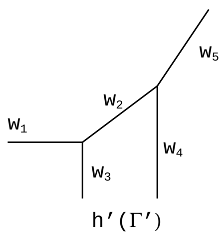

FIGURE 9.

Then we consider pre-log curves of type (Γ', h'). These curves have two components, and, as we saw in Subsection 5.1.3, each component is an image of a line in a projective plane under a covering map. In particular, each component is locally parametrized by the same parameters which parameterize the lines in the projective plane. Precisely speaking, as we noted in Subsection 5.1.3, there are several lines in $\mathbb{P}^2$ which project to the same line in the toric surface corresponding to the tropical curve.

Since each component has two parameters, we have four parameters in total. We remark that these parameters can be thought of as the value of affine coordinates of $\mathbb{P}^2$ at the intersections of the lines with the toric divisors. Noting this, we can take these parameters so that the incidence condition at the node corresponding to the bounded edge requires two of these parameters take the same value. So there are three free parameters. We write these parameters

$$(a, b, c) \in (\mathbb{C}^*)^3,$$

as in the case of weight one discussed above.

Now consider the toric degeneration defined respecting (Γ', h'). Pictorially, it is given as follows (Figure 10).

According to [9], there is a correspondence between pre-log curves of genus zero in X_0 of type (Γ', h') and torically transverse rational curves

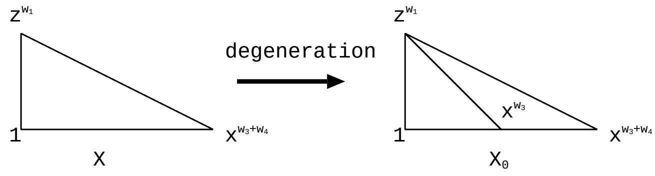

FIGURE 10.

in X which intersect the toric divisors with the multiplicities prescribed by the weights of the edges of $h'(\Gamma')$. In particular, a dense open subset of the set of these rational curves in X are parametrized by the parameters $(a, b, c) \in (\mathbb{C}^*)^3$ above.

Now, as in the standard smooth case above, suppose (Γ', h') is a part of genus one superabundant curve (Γ, h). One can calculate that, using appropriate coordinates as in Subsection 5.1.3, the lift of the component corresponding to (Γ', h') by the normal vector $x\partial_x$ induces a term

$$\frac{at^r}{bT} \cdot Z\partial_Z,$$

up to higher order terms in t, and a multiplicative constant which does not depend on the parameters. Here the exponent r is given as follows: if m is the weight of the edge of (Γ_0, h_0) which is the part of the path connecting (Γ_0, h_0) to the loop, then the integral length of the same edge is rm. T is an affine coordinate of the corresponding component of C_0 (cf. Subsection 5.1.3).

The parameterization of the torically transverse rational curves in X by parameters $(a, b, c) \in (\mathbb{C}^*)^3$ extends to a parametrization by $(a, b, c) \in \mathbb{C} \times (\mathbb{C}^*)^2$, so that the leading term $\frac{at^r}{bT} \cdot Z\partial_Z$ vanishes when the parameter a becomes zero, as in the standard case above. The rest of the argument is the same as the standard case.

Now we study the next example.

Example 2. This example also has a four valent vertex, but all the weights of the edges of the image are single integers. As before, first we assume all the weights w_i of the graph Γ_0 are one (including the bounded edge) and the associated toric variety is smooth.

As Example 1, we think that (Γ_0, h_0) is a part of a superabundant genus one curve (Γ, h). We define the affine subspace $A \subset N_{\mathbb{R}}$ as before, and assume some edge of $h_0(\Gamma_0)$ is contained in the connected component of $A \cap h(\Gamma)$ containing the loop, but the whole $h_0(\Gamma_0)$ is not contained in this component. In this case, there may be two edges of $h_0(\Gamma_0)$ contained in this component, and if this is the case, let $h(E_1) = \mathfrak{E}_1$ be the edge closer to the loop $h(L)$. If there is only one edge contained

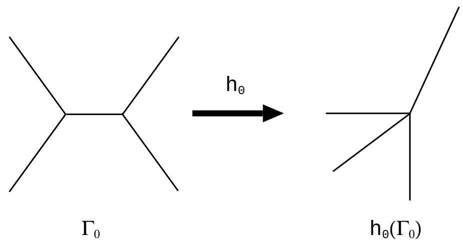

FIGURE 11.

in this component, then take it as $h(E_1) = \mathfrak{E}_1$. More accurately, there are following three cases:

(a) Two edges of $h_0(\Gamma_0)$ are contained in A.

(b) Only one edge of $h_0(\Gamma_0)$ is contained in A. Let $\mathfrak{E}_2, \mathfrak{E}_3, \mathfrak{E}_4$ be the other edges of $h_0(\Gamma_0)$, and let v_2, v_3, v_4 be the direction vectors of them. Then the dimension of the subspace of $N_{\mathbb{R}}$ spanned by $\bar{A}$ and $\{v_2, v_3, v_4\}$ is $\dim A + 1$. Here $\bar{A}$ is the linear subspace of $N_{\mathbb{R}}$ parallel to A.

(c) Only one edge of $h_0(\Gamma_0)$ is contained in A, and the dimension of the subspace spanned by $\bar{A}$ and $\{v_2, v_3, v_4\}$ is $\dim A + 2$.

We thus choose a bounded edge $h(E_1) = \mathfrak{E}_1$ in each case. As in Example 1, we take a coordinate system $\{x, y, z, w_1, \dots, w_{n-2}\}$ on the total space $\mathfrak{X}$ of a toric degeneration defined respecting (Γ, h). This time, we can assume the following: Namely, let $\varphi_0 : C_0 \to X_0$ be a generic pre-log curve of type (Γ, h). Let $\varphi_{0,v} : \mathbb{P}^1 \to X_0$ be the component corresponding to the vertex of (Γ_0, h_0). Then $\varphi_{0,v}(\mathbb{P}^1)$ is defined by the equations

$$ax + z + b = 0, \;\; ex + w_1 + f = 0, \;\; y = 0, \;\; w_2 = a_2, \dots, w_{n-2} = a_{n-2}.$$

Here $a, b, e, f, a_2, \dots, a_{n-2}$ are generic complex numbers. Also, the equation $xy = t$ holds, and the node p of C_0 corresponding to the edge E_1 is mapped into the set $\{x = 0\} \cap \{y = 0\}$. Clearly, $\varphi_{0,v}$ is parametrized by

$$(a, b, e, f; a_2, \dots, a_{n-2}) \in (\mathbb{C}^*)^4 \times (\mathbb{C}^*)^{n-3}.$$

This time, the fibers of tangent bundle of $\varphi_{0,v}(\mathbb{P}^1)$ are spanned by

$$x\partial_x - y\partial_y - a\frac{x}{z} \cdot z\partial_z - e\frac{x}{w_1} \cdot w_1\partial_{w_1} \tag{5}$$

as a subbundle of $\Theta_{\mathfrak{X}}$ near $\varphi_0(p)$.

Thus, when we extend the zeroth order lift of φ_0 given by $x\partial_x$, then the term

$$a\frac{t}{zy} \cdot z\partial_z + e\frac{t}{w_1 y} \cdot w_1\partial_{w_1}$$

appears on the neighboring component of $\varphi_0(C_0)$.

In the case (a) above, we can choose the coordinates so that one of the directions of $N_{\mathbb{R}}$ corresponding to $z\partial_z, w_1\partial_{w_1}$ (say, $z\partial_z$) is contained in

the subspace $\bar{A}$. Then only the term of a multiple of $w_1\partial_{w_1}$ contributes to the obstruction, and it does not vanish (unless there is another vertex, producing the obstruction in the direction of $w_1\partial_{w_1}$ and the integral distance to the loop is equal to that from the vertex of $h_0(\gamma_0)$ to the loop. But this does not happen in generic situations, see Lemma 48).

In the case (b), both terms of $a\frac{t}{zy}\cdot z\partial_z + e\frac{t}{w_1 y}\cdot w_1\partial_{w_1}$ contribute to the obstruction. Then it is easy to see that for any generic a, b, e, there is unique f such that the obstruction in the direction $z\partial_z \equiv w_1\partial_{w_1}$ mod $\bar{A}$ vanishes, if there is no other vertex which contributes to the obstruction in the direction $z\partial_z$ mod $\bar{A}$, and whose integral distance to the loop is shorter than to that from the vertex of $h_0(\gamma_0)$ to the loop.

In the case (c), the directions corresponding to $z\partial_z$ and $w_1\partial_{w_1}$ are linearly independent in $N_{\mathbb{R}}/\bar{A}$, so the contributions to the obstruction from the two terms do not cancel whatever the values of a, b, e and f are.

The general cases. We remark briefly about Example 2 with general weights and directions of the edges. Again, as in Subsection 5.1.3, the idea is considering the covering, reducing the argument to the standard case.

Consider pre-log curves corresponding to the tropical curve (Γ_0, h_0). As in the trivalent case, one can see that these curves are obtained as the image of the standard curves in $\mathbb{P}^3$ considered above, under a covering map

$$\mathbb{P}^3 \to \mathbb{P}_\Delta,$$

here $\mathbb{P}_\Delta$ is the toric variety constructed from the complete fan in $\mathbb{R}^3$ whose one dimensional fans are the rays spanned by the direction vectors of the unbounded edges of $h_0(\Gamma_0)$

In particular, these curves are also locally parametrized by $(\mathbb{C}^*)^4$, and the obstruction can be represented using these parameters, again as in the trivalent case. The rest of the argument is the same as Subsection 5.1.3 and the standard case above.

Summarizing, the cancelation of the obstruction contributed from the part corresponding to $h_0(\Gamma_0)$ occurs only in the case (b), when the curve $\varphi_0(\mathbb{P}^1)$ is in a special position (namely, the coefficients a, b, e, f satisfy one equation).

Remark 47. *(1) Using these examples, we can perform the calculation of the obstructions for pre-log curves of type* (Γ, h)*, where* (Γ, h) *is a tropical curve satisfying Assumption A, which is not necessarily immersive but may have several four-valent vertices.*

(2) *It is easy to see that, in this case too, as in Corollary 42, the leading term of the obstruction contributed from each vertex is determined by the configuration of the curve itself.*
(3) *It is also easy to see what happens when there are vertices of higher valence more than four. Namely, in Equations (4) and (5), higher order terms with respect to t, or terms of other directions (e.g., $w_2\partial_{w_2}, w_3\partial_{w_3}\dots$, or linear combinations of them) appear. However, these are irrelevant to the enumeration problem, see Lemma 48 below.*

Let us fix a degree Δ for a tropical curve. For genus one case, Δ determines the expected dimension of the moduli space (without referring to the data of the dimension of the ambient space). Namely, if (Γ, h) is a tropical curve of genus one of degree Δ, then the expected dimension of the moduli space is $|\Delta|$, the number of unbounded edges of Γ. We fix a generic affine constraint $\mathbf{A}$ of codimension $\mathbf{d}$ with $|\mathbf{d}| = |\Delta|$. (see Subsection 6.3).

Lemma 48. *Let (Γ, h) be a genus one superabundant tropical curve satisfying the generic constraint $\mathbf{A}$ (we do not a priori assume Assumption A). Assume the direction vectors of the edges of $h(\Gamma)$ span $N_{\mathbb{R}}$. Then if there is a pre-log curve of type (Γ, h) which is smoothable, the following conditions hold.*

(1) *$h(\Gamma)$ satisfies Assumption A.*
(2) *A vertex of $h(\Gamma)$ is at most four-valent.*
(3) *If $h(\Gamma)$ has a four-valent vertex, then it is locally isomorphic to Example 1, or* (b) *of Example 2.*

Remark 49. *(i) Thus, for enumeration problem of genus one curves, when $h(\Gamma)$ has a vertex which is four valent or more, it suffices to consider only the cases of Example 1 and case* (b) *of Example 2 above.*
(ii) For higher genus cases, as we noted in Remark 33 (iii), more degenerate case may appear. See Example 50.

Proof. As we noted above, for a genus one tropical curve (Γ, h), the expected dimension of the moduli space is the same as the number of unbounded edges of Γ. When we impose the condition to h that some bounded edges are contracted, then the freedom to deform the tropical curve decreases by the same number as the number of these bounded edges, if these edges are not contained in the loop. When an edge in the loop is contracted, it might force some other edges of the loop to be also contracted (see Example 50), but it does not affect the edges which are not contained in the loop.

Let $\varphi_0 : C_0 \to \mathfrak{X}$ be a pre-log curve of type (Γ, h). Let H be the dual obstruction space of Theorem 30. As in the calculation in Subsection 5.1 (see also Remark 47), for the vanishing of the obstructions, $(\dim H)$-dimensional conditions are imposed to the moduli space of the pre-log

curves which are perturbations of φ_0 (here we used the assumption that the direction vectors of the edges of $h(\Gamma)$ span $N_{\mathbb{R}}$). This, on the tropical curve side, implies the same dimensional conditions to the lengths of the edges of $h(\Gamma)$, *which are not contained in the loop*. So, contracting some edges of the loop imposes additional conditions, and we see that such curves do not satisfy generic incidence conditions. Thus, for tropical curves satisfying generic incidence conditions, we do not need to consider the case where some edges of the loop are contracted.

Let d_0 be the dimension of the moduli space of tropical curves which are deformations of (Γ, h). As we remarked above, the expected dimension of the moduli space of the tropical curves is given by $|\Delta|$. On the other hand, the actual dimension of the moduli space is

$$|\Delta| + \dim H.$$

Suppose (Γ, h) does not satisfy one of the conditions (1) to (3) of the lemma. Then, at least one of the following occurs:

(i) (Γ, h) does not satisfy Assumption A.
(ii) (Γ, h) satisfies Assumption A, but has a vertex of valence not less than five.
(iii) (Γ, h) satisfies Assumption A, but has a vertex of valence four, which is locally isomorphic to (a) or (c) of Example 2.

Each of these imposes a condition to h, which is independent of the conditions imposed by the vanishing of the obstruction.

So if the affine constraint $\mathbf{A}$ is generic, there is no tropical curve (Γ, h) of degree Δ which satisfies $\mathbf{A}$, $\dim H$ conditions for the vanishing of the obstruction, and one of (i) to (iii) above. Thus, the tropical curve (Γ, h) satisfies the conditions (1) to (3) of the lemma. $\square$

Example 50. Here we consider an example of a higher genus tropical curve, in which a situation where

- Assumption A does not hold and
- highly degenerate tropical curves inevitably appear for the enumeration problem

happens. Namely, consider the following Figure 12.

It is an immersed genus two superabundant curve in $\mathbb{R}^3$, and $\dim H$ is equal to one, where H is the dual space of the obstructions. The obstruction is localized at the loop on the left, drawn by bold lines. The loop on the right is, as a genus one tropical curve, non-superabundant. Assume that the integral length of the edge E_2 is twice of that of E_1. Since the loop containing E_i is non-superabundant (and it has only four edges), one sees that this ratio of the integral lengths of E_2 and E_1 does not change when we deform the tropical curve.

Consider a condition for the smoothability of this tropical curve. Although it is not genus one, we can calculate the obstruction as in the

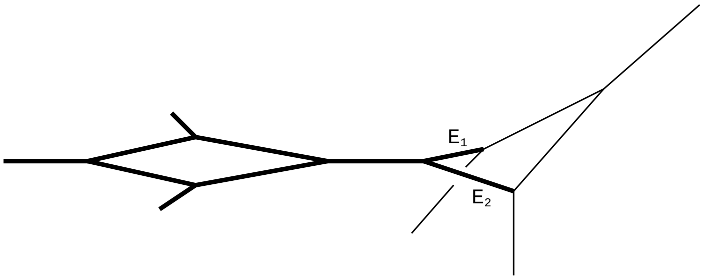

FIGURE 12. Segments drawn by bold lines are contained in the affine subplane spanned by the loop.

calculation in this section, and it is easy to see that, for the vanishing of the obstruction, the integral lengths of the edges E_1 and E_2 should be equal. However, according to the above observation, it only happens when both E_1 and E_2 are contracted. This already breaks Assumption A. Moreover, it is easy to see that when the edge E_1 or E_2 is contracted, the whole loop must be contracted. Thus, for higher genus curves, when we try to remove Assumption A, we cannot moderately weaken it, but we have to consider much more general situations.

Due to Lemma 48, we formulate the generalization of Theorem 45 only for (Γ, h) with at most four-valent vertices. It is straightforward to extend it to more general genus one curves (at least when (Γ, h) satisfies Assumption A).

Let (Γ, h) be a genus one superabundant tropical curve satisfying Assumption A and assume that a vertex of $h(\Gamma)$ is at most four-valent. We use the same notations Γ', $\mathcal{H}$, $\Gamma_{\mathcal{H}}$, $p_i^{\mathcal{H}}, \mathcal{P}_i$ as in Definition 44.

Definition 51. (Γ, h) is said to be *well-spaced* if one of the following condition is satisfied for any affine hyperplane $\mathcal{H}$ containing $h(\Gamma')$.

(1) The set $\{\ell_{(\Gamma,h)}(\mathcal{P}_1), \dots, \ell_{(\Gamma,h)}(\mathcal{P}_j)\}$ contains at least two minimum.
(2) The set $\{\ell_{(\Gamma,h)}(\mathcal{P}_1), \dots, \ell_{(\Gamma,h)}(\mathcal{P}_j)\}$ contains only one minimum. Let $p_i^{\mathcal{H}}$ be the vertex of $h(\Gamma_{\mathcal{H}})$ at which $\ell_{(\Gamma,h)}(\mathcal{P}_i)$ takes the minimum among $\{\ell_{(\Gamma,h)}(\mathcal{P}_1), \dots, \ell_{(\Gamma,h)}(\mathcal{P}_j)\}$. Then $p_i^{\mathcal{H}}$ is four valent, and near $p_i^{\mathcal{H}}$, (Γ, h) is locally isomorphic to Example 1 or (b) of Example 2 above.

The next theorem follows from an obvious extension of the calculation in Subsection 5.1.

Theorem 52. *Let (Γ, h) be a genus one tropical curve satisfying Assumption A and assume that each vertex of $h(\Gamma)$ is at most four-valent. Then it is smoothable if and only if it is well-spaced.* □

5.2.1. **Examples.** Here we give several examples in the case of genus one cubic curves in $\mathbb{P}^3$. The most standard superabundant case, which

is the subject of Speyer's original well-spacedness condition [12], is given in the following figure (Figure 13).

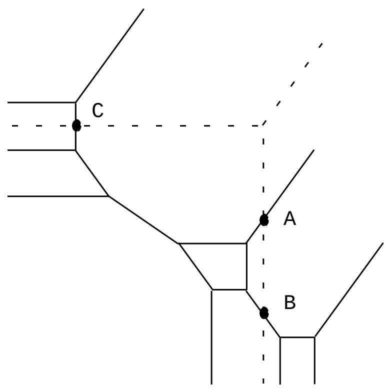

FIGURE 13.

There are three unbounded edges for each of the following directions

$$(-1, 0, 0), \quad (0, -1, 0), \quad (1, 1, 1), \quad (0, 0, -1).$$

All of these edges are weight one. The three black dots in the figure means the unbounded edges of direction $(0, 0, -1)$. Genus one cubic curves in $\mathbb{P}^3$ are known to be contained in some projective plane, and it is true also in the tropical case. The dotted lines represent the image of the one dimensional skeleton of the tropical hyperplane containing the genus one tropical cubic curve under the projection to the plane.

In this figure, the vertices A, B, C are the one-valent vertices of Γ', and the vertices A and B assure the well-spacedness condition.

When we slide the curve on the plane, the picture becomes as follows (Figure 14).

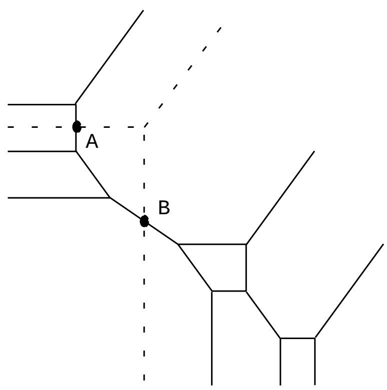

FIGURE 14.

In this case, the unbounded edge of direction $(0, 0, -1)$ at the vertex B has weight two, and this is the case of Example 1. So this satisfies the extended well-spacedness condition of Definition 51.

Next, consider the cases when some of the horizontal unbounded edges of Γ are merged, as in the following picture (Figure 15). Note that the unbounded edges are merged *in* Γ. That is, the image of the merged edge has weight two, not (1, 1) (the latter case is not generic in the space of smoothable tropical curves).

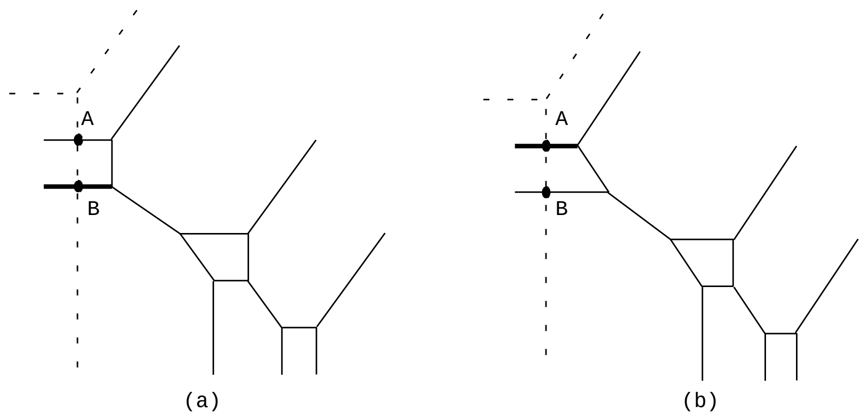

FIGURE 15.

As curves on a projective plane, these are cubic curves with the condition that they have one intersection point of multiplicity two with a toric divisor. The bold lines have weight two, and correspond to these intersection points.

On the other hand, the vertical unbounded edges emanating from the vertices on the bold lines (B of the figure (a) and A of the figure (b)) also have total additive weight two (Definition 14).

In the case of (a), the integral distance from B to the loop is shorter than that from A, so for the well-spacedness condition, it is necessary that the vertical edge from B has weight $(1, 1)$. This is the case of Example 1.

In the case of (b), the integral distance from A or B to the loop is the same. Since the length of the edge with weight two should be halved, the leading contribution to the obstruction comes from the vertex A. There are two cases.

(i) The vertical edge from A has weight (1, 1).
(ii) The vertical edge from A has weight two. In this case, the tropical curve is immersed.

The former case is smoothable (Example 1). The latter case corresponds to a genus one cubic curve which has two intersection points of multiplicity two with toric divisors, but not smoothable.

Note that not every tropical curve is smoothable, even if it is contained in a tropical hyperplane. Examples are given in the following figures (Figure 16).

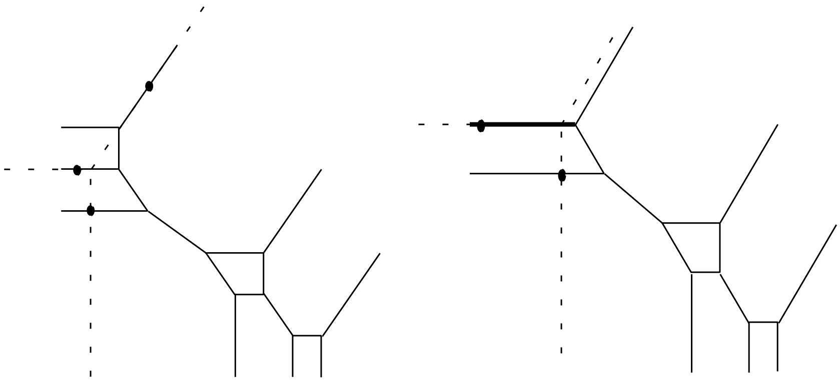

FIGURE 16.

Finally, we give an example that clearly] shows why we have to divide the integral length of an edge by its weight.

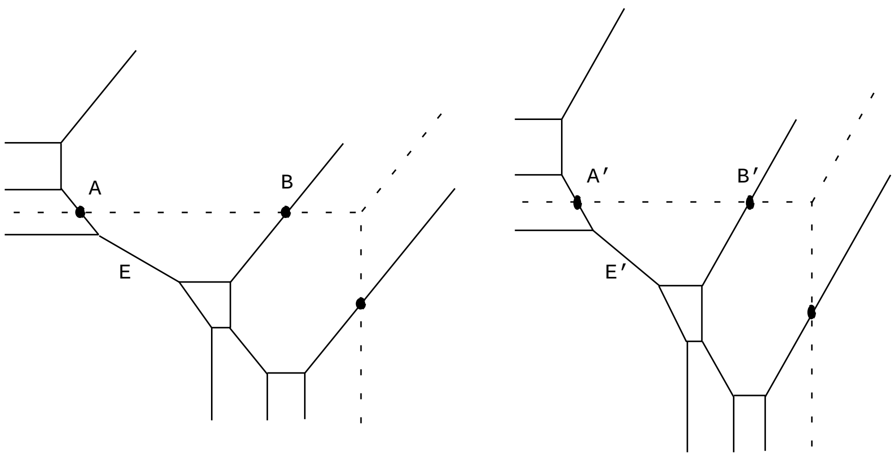

FIGURE 17.

The left tropical curve corresponds a standard degree three curve in $\mathbb{P}^3$. The right tropical curve is obtained from the left by the linear transformation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ on the plane. As a result, this tropical curve is also smoothable. Note that the edge E' has the weight two.

In the left tropical curve, the integral lengths of the paths from the vertices A and B to the loop are the shortest ones. While, the integral length of the edge E' in the right tropical curve is the twice of that of the edge E in the left tropical curve. So the integral length of the path from the vertex A' to the loop is greater than that of the path from the vertex B' to the loop. However, when we halve the integral length of E', these two are the same, adjusting to the well-spacedness condition.

6. Correspondence theorem for superabundant curves II: Enumerative result for genus one curves

6.1. **Kuranishi map.** Now we begin to study the enumeration problems for superabundant genus one curves. We use the same notations as in the previous section. In this subsection, we work under the following assumption, due to Lemma 48.

Assumption C. (Γ, h) satisfies Assumption A, and the vertices are at most-four valent. Four valent vertices are locally isomorphic to the one in Example 1 or (b) of Example 2 in the previous subsection.

We saw in Section 5, assuming that the tropical curve (Γ, h) satisfies Assumption C, when (Γ, h) is well-spaced, there is a pre-log curve of type (Γ, h) which can be smoothed. In this subsection, we study the local behavior of the moduli space of these smoothings.

First we review the situation of the non-superabundant case. The corresponding local result in genus zero case (which can be extended to general non-superabundant case) was given in Lemma 7.2 of [9]. The result is that the smoothings are locally parametrized by $H^0(\mathcal{N}_{C_0/X_0})$.

In general, the actual moduli space is represented as the inverse image of zero of the *Kurainshi map*

$$\mathcal{K} : H^0(\mathcal{N}_{C_0/X_0}) \to H^1(\mathcal{N}_{C_0/X_0})$$

(we give the precise definition soon later). In the non-superabundant case, the space $H^1(\mathcal{N}_{C_0/X_0})$ is zero, so locally $H^0(\mathcal{N}_{C_0/X_0})$ itself can be thought of as the moduli space.

Let (Γ, h) be a well-spaced superabundant tropical curve satisfying Assumption C and $\varphi_0 : C_0 \to X_0$ be a pre-log curve of type (Γ, h) which can be smoothed, which exists by Theorem 52. In our case, we can describe the local behavior of the Kuranishi map $\mathcal{K}$ around C_0 by the calculation in Section 5. Namely, assume that we have a k-th order smoothing

$$\varphi_k : C_k \to \mathfrak{X}$$

of φ_0, which exists by the smoothability assumption on φ_0, where C_k is a k-th order smoothing of C_0. Other smoothings are parametrized by the elements of some subset of

$$H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k.$$

Our purpose is to describe the condition for an element $\mathfrak{n}$ of $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ to be contained in this subset.

We take a basis $\{\mathfrak{a}_1, \dots, \mathfrak{a}_a\}$ of $(\bar{A})^\perp$, here $a = \dim(\bar{A})^\perp$. Then we define the Kuranishi map in the following way. Note that as we saw in Step 5 in Subsection 5.1, the obstructions $o(\mathfrak{n}; \varphi_k)_i$ (Definition 43), which are given as $N_{\mathbb{C}}$-valued germs of rational sections, and elements

in $H \cong (\bar{A})^{\perp}$ make a natural pairing. Let us write it as

$$\langle \mathfrak{a}, o \rangle,$$

here $\mathfrak{a} \in H$ and o is a germ of a rational section.

Definition 53. We define the *Kuranishi map of order k at φ_k*

$$\mathcal{K} : H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k \to (\mathbb{C}[t]/t^k)^a$$

by

$$\mathfrak{n} \mapsto \{\langle \mathfrak{a}_j, \sum_i o(\mathfrak{n}; \varphi_k)_i \rangle\}_{j=1,\dots,a},$$

here the index i parametrizes the set $\{\mathcal{U}_i\}$ of components of $\Gamma \setminus L$.

By definition, we have the following. As usual, we assume that the direction vectors of the edges of $h(\Gamma)$ span $N_{\mathbb{R}}$.

Proposition 54. *A section $\mathfrak{n}$ corresponds to a k-th order smoothing of φ_0 if and only if $\mathcal{K}(\mathfrak{n}) = 0$.* □

The map $o(\mathfrak{n}; \varphi_k)_i$ is not affine linear in $\mathfrak{n}$ nor the coefficients of the defining equations of $\varphi_k(C_k)$, and hard to compute in general. However, for the leading terms, we have better understanding, as we saw in Proposition 41. In particular, we can see from the proof of Theorem 45 the following result. Take a linear coordinate system $\{x_1, \dots, x_b\}$ on $H^0(\mathcal{N}_{C_0/X_0})$ and represent a vector $\mathfrak{n}$ in $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ by

$$x_i(\mathfrak{n}) = x_{0,i}(\mathfrak{n}) + x_{1,i}(\mathfrak{n})t + \cdots + x_{k-1,i}(\mathfrak{n})t^{k-1}, \ \ i = 1, \dots, b, \ \ x_{j,i}(\mathfrak{n}) \in \mathbb{C}.$$

Proposition 55. *Assume k is sufficiently large. We can choose the basis $\{\mathfrak{a}_1, \dots, \mathfrak{a}_a\}$ so that the space of solutions $\mathcal{K}^{-1}(0)$ is a perturbation of a linear subspace of codimension a in $H^0(\mathcal{N}_{C_0/X_0})$ in the following sense:*

- *The set*

  $$E = \{\{x_{0,i}(\mathfrak{n})\}_{i=1,\dots,b} |\ \mathcal{K}(x_1(\mathfrak{n}), \dots, x_b(\mathfrak{n})) = 0\}$$

  is a linear subspace of

  $$H^0(\mathcal{N}_{C_0/X_0}) \cong \mathbb{C}^b = \{\{x_{0,i}\}_{i=1,\dots,b} | x_{0,i} \in \mathbb{C}\}$$

  of codimension a.
- *Let*

  $$h_1(x_i) = h_1(x_{0,i}) = 0, \dots, h_a(x_i) = h_a(x_{0,i}) = 0$$

  be a set of linear equations on $H^0(\mathcal{N}_{C_0/X_0})$ defining E. Then the defining equations for the set $\mathcal{K}^{-1}(0) \subset H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ are written in the form

  $$h_1(x_i) + tg_1(x_i), \dots, h_a(x_i) + tg_a(x_i).$$
- *The polynomials $h_1, \dots, h_a$ depend only on φ_0, and do not depend on k, if k is large enough.*

Proof. First note that $\mathcal{K}(0) = 0$, since $\mathfrak{n} = 0$ corresponds to the smoothing φ_k. Write

$$\mathcal{K}(x_1, \dots, x_b) = (\mathcal{K}_1(x_1, \dots, x_b), \dots, \mathcal{K}_a(x_1, \dots, x_b))\,.$$

Now we explain how to determine the basis $\{\mathfrak{a}_1, \dots, \mathfrak{a}_a\}$ of $(\bar{A})^\perp$. We take $\mathfrak{a}_1$ to be a generic vector in $(\bar{A})^\perp$. Then as in the argument at the beginning of Step 6 in the previous section, we determine an affine subspace A_1 of $N_{\mathbb{R}}$ and its parallel linear subspace $\bar{A}_1$, using one of the one-valent vertices of Γ' which has the minimal integral distance to the loop L. Note that by definition $\bar{A}_1$ contains the subspace $\bar{A}$. Now define $\mathfrak{a}_2$ to be a generic vector in $(\bar{A}_1)^\perp$. If there is another one-valent vertex of Γ' such that

- the integral distance to the loop is minimal, and
- the two dimensional subspace of $N_{\mathbb{R}}$ spanned by the direction vectors of the edges emanating from this vertex is not contained in $\bar{A}_1$,

then let $\bar{A}_2$ to be the span of $\bar{A}_1$ and this two dimensional subspace. Then let $\mathfrak{a}_3$ be a generic vector in $(\bar{A}_2)^\perp$. Continue this until there is no one-valent vertex of Γ' satisfying the above two conditions.

Let A_c be the affine subspace obtained at the last step of this process, and $\bar{A}_c$ be the parallel linear subspace. Let Γ'_{A_c} be the connected component of $h(\Gamma) \cap A_c$ containing the loop of $h(\Gamma)$. Then consider the one-valent vertices of Γ'_{A_c} and do the same process as above.

Continuing this, since the direction vectors of the edges of $h(\Gamma)$ span $N_{\mathbb{R}}$, one eventually determines a basis of $(\bar{A})^\perp$.

Using this basis, it is easy to see that $\mathcal{K}$ has the following form for large enough k. Namely,

$$\begin{aligned}\mathcal{K}_i(x_1, \dots, x_b) = \ & h_i(x_{0,1}, \dots, x_{0,b})t^{L_i} \\ & + f_{i,1}(x_{0,1}, \dots, x_{0,b}, x_{1,1}, \dots, x_{1,b})t^{L_i+1} \\ & + f_{i,2}(x_{0,1}, \dots, x_{0,b}, x_{1,1}, \dots, x_{1,b}, x_{2,1}, \dots, x_{2,b})t^{L_i+2} \\ & + \cdots\,.\end{aligned}$$

Here L_i is a positive integer such that

$$L_1 \le L_2 \le \cdots \le L_a,$$

and $h_i, f_{i,j}$ are polynomials. Terms of h_i are given as follows. Namely, each of the one-velent vertices we used to define $\mathfrak{a}_i$ gives a contribution to the obstruction, whose leading term is of the form $(\frac{k_0}{m_0}) \cdot (\frac{l_1}{m_1}) \cdot (\frac{k_1}{m_1}) \cdot \cdots \cdot (\frac{l_{N-1}}{m_{N-1}}) \cdot (\frac{k_{N-1}}{m_{N-1}}) \cdot (\frac{l_N}{m_N}) \cdot t^M$, with the data of direction in $N_{\mathbb{R}}$, as we calculated in Proposition 41. The numbers k_i, l_i, m_i are determined by the data of φ_0. The sum of these leading terms is zero, since φ_0 is smoothable. A section $\mathfrak{n}$ perturbs the numbers k_i, l_i, m_i by adding

$$t \cdot P(x_{0,1}, \dots, x_{0,b}) + \text{terms of higher order in } t,$$

here P is linear homogeneous in $x_{0,i}$. Substituting these perturbed values of k_i, l_i, m_i to $(\frac{k_0}{m_0})\cdot(\frac{l_1}{m_1})\cdot(\frac{k_1}{m_1})\cdot\dots\cdot(\frac{l_{N-1}}{m_{N-1}})\cdot(\frac{k_{N-1}}{m_{N-1}})\cdot(\frac{l_N}{m_N})$, and expanding it with respect to t, the terms of h_i are given as the coefficients of t. Clearly, h_i is linear homogeneous in $x_{0,i}$, and does not depend on $x_{j,i}$, $j \geq 1$. Note that the integer L_i equals M_i+1, where M_i is the length of the path from $\mathfrak{a}_i$ to the loop L in the sense of Definition 38.

By the assumption that the direction vectors of the edges of Γ span $N_{\mathbb{R}}$, $\{h_i\} : \mathbb{C}^b \cong H^0(\mathcal{N}_{C_0/X_0}) \to \mathbb{C}^a$ is surjective. So it defines a subspace of $H^0(\mathcal{N}_{C_0/X_0})$ of codimension a. The proposition follows from this and the above form of $\mathcal{K}$. $\square$

This can be thought of as an analogue of the local description of the moduli space, Lemma 7.2 of [9], and also of the transversality result for the moduli space, Proposition 7.3 of [9], replacing the incidence conditions there by the obstruction.

6.2. **Geometric interpretation.** Let (Γ, h) be a genus one superabundant tropical curve satisfying Assumption C, and $\varphi_0 : C_0 \to X_0$ be a pre-log curve of type (Γ, h). We have seen that if (Γ, h) satisfies the well-spacedness condition, then there exists φ_0 which can be smoothed to any order in t (Theorems 45, 52), and at each such a pre-log curve, the solution space of the Kuranishi map is modeled on a linear subspace of $H^0(\mathcal{N}_{C_0/X_0})$ of codimension $a = \dim H$ (Proposition 55). In particular, the moduli space of smoothing is smooth in appropriate sense.

The rest of the problem for describing the moduli space is, to understand how it is written down using the data we have. In this subsection we give a description of the moduli space using the data of the tropical curve (Γ, h).

Let (Γ, h) be a genus one superabundant tropical curve as above. Assume again that $\dim H = 1$ for simplicity. Then, for the vanishing of the obstruction, essentially it suffices to look only at the one-valent vertices of Γ' whose distances to the loop are minimal. Let $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ be the set of those one-valent vertices. Given a pre-log curve $\varphi_0 : C_0 \to X_0$ of type (Γ, h), the leading term of the obstruction contributed from each component of C_0 corresponding to the vertices α_i is calculated in Proposition 41. Now we study the meaning of

$$R = \left(\frac{k_0}{m_0}\right)\cdot\left(\frac{l_1}{m_1}\right)\cdot\left(\frac{k_1}{m_1}\right)\cdot\dots\cdot\left(\frac{l_{N-1}}{m_{N-1}}\right)\cdot\left(\frac{k_{N-1}}{m_{N-1}}\right)\cdot\left(\frac{l_N}{m_N}\right)$$

in terms of tropical geometry. For general cases when $\dim H$ is not necessarily one, contributions to the leading terms of the obstruction may come from the vertices in $\Gamma \setminus \Gamma'$, as discussed in Step 6 in the previous section. However, the computation of the leading terms of

the obstruction contributed from these vertices is completely the same as what we did so far, and the discussion below applies to them as well.

6.2.1. *Review of tropicalization.* Recall the *tropicalization* of a complex curve in $(\mathbb{C}^*)^2$ from [4]. Let

$$Log : (\mathbb{C}^*)^2 \to \mathbb{R}^2$$

be a map defined by

$$(x, y) \mapsto (\log |x|, \log |y|),$$

and let $H_\tau : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$, $\tau > 1$, be a map defined by

$$(x, y) \mapsto (|x|^{\frac{1}{\log \tau}} \frac{x}{|x|}, |y|^{\frac{1}{\log \tau}} \frac{y}{|y|}).$$

Let us define

$$Log_\tau = Log \circ H_\tau : (\mathbb{C}^*)^2 \to \mathbb{R}^2.$$

The tropicalization of a complex curve $Q \subset (\mathbb{C}^*)^2$ is given by the limit

$$\lim_{\tau \to \infty} Log_\tau(Q).$$

For a fixed Q, the result is a union of rays emanating from the origin. To obtain more non-trivial result, one considers a curve whose defining equation contains powers of τ in its coefficients, and takes the above limit.

In our case, we consider a family of lines L_τ:

$$\tau^a x + \tau^b y + 1 = 0,$$

here a, b are real numbers. Mikhalkin [4], Lemma 8.3 and its proof, shows that the tropicalization of L_τ is the tropical curve defined by the tropical polynomial

$$\max\{a + x, b + y, 0\}$$

(in this particular case, it is easy to prove it directly). This is the union of rays m_x, m_y parallel to the x- and y- axes, and another ray parallel to the line $\{x = y\}$, all emanating from a unique vertex v.

Conversely, suppose that we are given a standard tropical line in $\mathbb{R}^2$ whose vertex is at $(-a, -b)$, that is, the tropical line given by the tropical polynomial $\max\{a + x, b + y, 0\}$ above. These curves are parametrized by the position of the vertex, so the moduli space $\mathcal{M}$ is isomorphic to $\mathbb{R}^2$. This isomorphism is fixed once we determine the base point $(-a, -b)$. There is an obvious bijection between the set of torically transverse lines in $\mathbb{P}^2$ and the 'complexification' of $\mathcal{M}$, $\mathcal{M}_\mathbb{C} = \mathcal{M} \times (S^1)^2 \cong (\mathbb{C}^*)^2$, for each $\tau > 1$. Namely, for a complex line $kx + ly + m = 0$, the corresponding point in $\mathcal{M}_\mathbb{C}$ is

$$\left(\frac{1}{\log \tau} e^{i(\arg k - \arg m)} \cdot \log \left|\frac{k}{m}\right| - a, \ \frac{1}{\log \tau} e^{i(\arg l - \arg m)} \cdot \log \left|\frac{l}{m}\right| - b\right).$$

Taking the real part, we obtain a point

$$\left(\frac{1}{\log \tau} \cdot \log \left|\frac{k}{m}\right| - a,\ \ \frac{1}{\log \tau} \cdot \log \left|\frac{l}{m}\right| - b\right).$$

in $\mathcal{M}$, giving the corresponding tropical line. We call this a *tropicalization with modulus* τ.

Although this correspondence depends on the choice of the positive real number τ, there is a natural map between them for different τ, once we fix a curve, for example, $x+y+1=0$, as a base point on the side of moduli of holomorphic lines, and take $(0,0)$ as a base point for $\mathcal{M} \cong \mathbb{R}^2$. Namely, the map between τ and τ' is given by magnification by $\frac{\log \tau}{\log \tau'}$.

For convenience, we extend the definition of path length of tropical curves defined over integers (Definition 38) to tropical curves defined over rational numbers. Let (Γ, h) be any tropical curve satisfying Assumption A. Let E be an edge of Γ and $\mathfrak{E} = h(E)$ be its image. Let $\vec{v}_{\mathfrak{E}}$ be the primitive integral generator of $\mathfrak{E}$ (one may choose the sign arbitrarily, it does not affect the definition). We take a standard integral basis of $N_{\mathbb{R}} = \mathbb{Z}^n \otimes \mathbb{R}$, and fix a standard Euclidean norm $\|\cdot\|$ on the vectors in $N_{\mathbb{R}}$ and the associated distance function $d(\cdot,\cdot)$ on $N_{\mathbb{R}}$.

Definition 56. Let x, y be two points on $\mathfrak{E}$ and assume $\mathfrak{E}$ is not an unbounded edge. Let w be the weight of the edge $\mathfrak{E}$. Note that by Assumption A, when $\mathfrak{E}$ is not unbounded, then its weight is a single integer. Then we define the length of the segment $\overline{xy}$ by

$$d_{(\Gamma,h)}(\overline{xy}) = \frac{d(x,y)}{w \parallel \vec{v}_{\mathfrak{E}} \parallel}.$$

If $\mathcal{P}$ is a path on $h(\Gamma)$, we define its length $d_{(\Gamma,h)}(\mathcal{P})$ by the obvious extension of this.

This is a modification of the distance function used by Mikhalkin and Zhalkov in their paper [6]. Clearly, when the ends of $\mathcal{P}$ are lattice points, then $d_{(\Gamma,h)}(\mathcal{P})$ is nothing but the length $\ell_{(\Gamma,h)}(\mathcal{P})$ of $\mathcal{P}$.

6.2.2. *Cases of two vertices.* For later use, we extend the above consideration to the case of trivalent tropical curves with two vertices. There are two cases:

(1) The tropical curve is contained in an affine plane.
(2) The tropical curve is not contained in a plane, but in a three dimensional subspace.

First we consider the case (2). We consider an immersive tropical curve whose type is given by (Γ_0, h) in the figure below.

Here all the edges have weight one, and the directions of the edges E_1, E_2 and E_3 are $(-1,0,0), (0,-1,0)$ and $(0,0,-1)$, respectively.

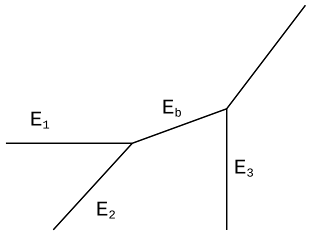

FIGURE 18.

Although general tropicalization is argued only for two dimensional ambient spaces in [4], it is easy to extend it to three dimensional ambient spaces in these particularly simple cases. The corresponding holomorphic curves are lines in $\mathbb{P}^3$, and they are given by equations

$$\alpha x + \beta y + 1 = 0, \quad \gamma x + \delta z + 1 = 0,$$

Replacing $\alpha, \beta, \gamma, \delta$ by $\tau^a, \tau^b, \tau^c, \tau^d$ with $a, b, c, d \in \mathbb{R}$, we perform tropicalization. Then one easily sees that the vertices are given by

$$(-a, -b, -d), \quad (-c, -b, -d),$$

where we need to impose $a > c$ so that the tropical curve has the above combinatorial type. In general, the combinatorial type of the tropical curve depends of the values of a and c. However, whatever the values of a and c are (so the combinatorial type may change), the length of the bounded edge E_b (in the sense of Definition 56) is $|a - c|$.

Any trivalent immersive tropical curve with two vertices which is not contained in an affine plane is obtained by a linear transformation of these standard tropical curves. When such a tropical curve (Γ, h), whose bounded edge has weight w, is obtained by transforming the standard tropical curve above, it is easy to see that the bounded edge has length $w|a - c|$.

As in the one-vertex case discussed above, these tropicalization can be lifted to the version parametrized by τ. Namely, to the lines in $\mathbb{P}^3$ given by the equations above, we assign a tropical curve whose vertices are

$$(-\frac{1}{\log \tau} \log |\alpha|, -\frac{1}{\log \tau} \log |\beta|, -\frac{1}{\log \tau} \log |\delta|),$$
$$(-\frac{1}{\log \tau} \log |\gamma|, -\frac{1}{\log \tau} \log |\beta|, -\frac{1}{\log \tau} \log |\delta|).$$

The length of the bounded edge depends on τ, but when we fix a curve as a base point (in the above case, the curve $x + y + 1 = 0, x + z + 1 = 0$ for example) of the moduli, there are natural maps between these tropicalizations.

On the other hand, the length of the bounded edge depends only on the values of $\frac{\alpha}{\gamma}$ and w, or, the value of $|\frac{\alpha}{\gamma}|^w$ (or $|\frac{\gamma}{\alpha}|^w$. We should take the one which is larger than one so that the corresponding edge

has positive length.), aside from τ. In the main text, the quantity $|\frac{\alpha}{\gamma}|^w$ corresponds to

$$\left|\frac{k_{i-1}}{m_{i-1}}\right|^{w_i} \cdot \left|\frac{l_i}{m_i}\right|^{w_i}.$$

Remark 57. *Here, we are considering a tropicalization of pre-log rational curves which have four intersections with the toric divisors, so that the corresponding tropical curve is, generically, has two trivalent vertices. While in the main text, tropical curves are (when it is immersive) combinatorial counter part of maximally degenerate curves, whose components intersect the toric divisor at three points. In the latter case, the tropicalization of each component has no bounded edge and we cannot discuss about the length.*

However, according to [9] *(see also Subsection 5.2), there is a natural correspondence between generic pre-log rational curves which have four intersections with the toric divisors, and maximally degenerate pre-log curves with two components. Using this correspondence, we can talk about the length of the bounded edges of the tropicalization of the maximally degenerate curves, and the above relation between* $|\frac{\alpha}{\gamma}|^w$ *and* $\left|\frac{k_{i-1}}{m_{i-1}}\right|^{w_i} \cdot \left|\frac{l_i}{m_i}\right|^{w_i}$ *is deduced from this correspondence.*

Note that the value $\left|\frac{k_{i-1}}{m_{i-1}}\right|^{w_i} \cdot \left|\frac{l_i}{m_i}\right|^{w_i}$ is always larger than one by construction (otherwise, it causes a contradiction geometrically). Then, the length of the bounded edge in the sense of Definition 56 is given by

$$\frac{1}{w_i} \cdot \frac{1}{\log \tau} \cdot \left(\log \left|\frac{k_{i-1}}{m_{i-1}}\right|^{w_i} \cdot \left|\frac{l_i}{m_i}\right|^{w_i}\right) = \frac{1}{\log \tau} \cdot \left(\log \left|\frac{k_{i-1}}{m_{i-1}}\right| + \log \left|\frac{l_i}{m_i}\right|\right).$$

Next, let us consider the case (1), that is, when the tropical curve is contained in an affine plane. Note that these curves can be obtained by projecting appropriate curves of the case (2) above by a projection

$$\mathbb{Z}^3 \otimes \mathbb{R} \to \mathbb{Z}^2 \otimes \mathbb{R}$$

to the first two factors. Note that this preserves the integral length (so the length of Definition 56) of the bounded edge. In general, it is hard to write down the defining equation of the projected curve, however, in Subsection 5.1.3, we parametrized these curves by their coverings, and such a covering is obtained by the composition of the linear transformation from the standard tropical curve considered above, and the projection. So, the local moduli parameters are just the coefficients $\alpha, \beta, \gamma, \delta$ above, and the correspondence between $|\frac{\alpha}{\gamma}|^w$ and $\left|\frac{k_{i-1}}{m_{i-1}}\right|^{w_i} \cdot \left|\frac{l_i}{m_i}\right|^{w_i}$ given in the case (2) is valid in this case, too.

Summarizing, we have the following. we use the same notation as in Section 5 (see, in particular, Subsections 5.1.2 and 5.1.3).

Proposition 58. *Let (Γ, h) be an immersive superabundant genus one curve, and take a path $\mathcal{P}$ connecting a one valent vertex of Γ' and the loop. Consider a pre-log curve $\varphi_0 : C_0 \to X_0$ of type (Γ, h). Let Γ_i be the tropical curve with two vertices α_i, α_{i+1} of $\mathcal{P}$, obtained by taking a neighborhood of the edge connecting α_i and α_{i+1}. Then, under the tropicalizaion with modulus τ, the bounded edge corresponding to the node between the components ℓ_{α_i} and $\ell_{\alpha_{i+1}}$ of C_0 has length*

$$\frac{1}{\log \tau}\left(\log\left|\frac{k_{i-1}}{m_{i-1}}\right| + \log\left|\frac{k_i}{m_i}\right|\right).$$

□

Remark 59. *Note that the bounded edges of the tropical curve (Γ, h) has its own length. The length of the bounded edges of the tropicalization ($\frac{1}{\log \tau}\cdot\left(\log\left|\frac{k_{i-1}}{m_{i-1}}\right| + \log\left|\frac{k_i}{m_i}\right|\right)$ in the above proposition) is apriori unrelated to the length of the corresponding edge of (Γ, h). In terms of Remark 60, the length of the edges of (Γ, h) is related to the* macroscopic *aspect of the tropical curve, while the length of the tropicalization is related to the* microscopic *aspect of it.*

6.2.3. **Immersive case.** Here we assume (Γ, h) is immersive, and deal with more general cases later. Before stating the next result, we note the following remark.

Remark 60. *In our study of the correspondence between tropical curves and holomorphic curves, tropical curves play two roles, (1) "macroscopic" and (2) "microscopic", so to speak.*

(1) First, a tropical curve is considered as a part of a polyhedral decomposition of $N_{\mathbb{R}}$, determining the degeneration of a toric variety. In this interpretation, the integral lengths of the edges correspond to the order of t. The well-spacedness condition is deduced from this, and is enough for proving the existence of at least one smoothing.

(2) Second, tropical curves are considered as the combinatorial counterpart of pre-log curves in X_0. In this case, the integral lengths of the edges correspond to the moduli of the pre-log curves. Of course, the order of t is also related to the moduli, but this side looks more refined data (fixing the order of t). In this subsection, we are studying this side.

Clearly, these two interpretations are mutually related. This can be particularly obserbed in Theorems 62, 63 below.

In [9]*, since there was no need to consider the order of t, these two roles were not specifically distinguished.*

Now let us interpret the vanishing of the obstruction from tropical geometry. Again, in the main text we assume that $\dim H = 1$ for

the simplicity of the explanation. When $\dim H$ is larger than one, we argue inductively as in Step 6 of Section 5. Recall we are assuming (Γ, h) is immersive (see Theorem 63 for the non-immersive case). As we mentioned in Remark 60, in this subsection the tropical curves are the combinatorial counterpart of the holomorphic curves in X_0. In particular, the degeneration $\mathfrak{X}$ is already fixed so that there are at least two components of $\varphi_0(C_0)$ corresponding to one-valent vertices of Γ' which have the following properties:

- We can calculate the leading terms of the obstructions contributed from these components, as in Section 5. These terms have the data of the order of t, and the orders of the obstructions contributed from these components are the same.
- This order of t is minimal among the obstructions contributed from the components of $\varphi(C_0)$ corresponding to the one-valent vertices of Γ'.

In fact, generically there are only two such vertices, since the set of those tropical curves which have more than two vertices with the properties above is contained in a lower dimensional subset in the set of tropical curves satisfying the well-spacedness condition. So we assume that there are just two such vertices. Let $\mathcal{P}$, $\mathcal{P}'$ be the paths from these vertices to the loop.

Then the leading terms of the obstruction contributed from these vertices are, according to Proposition 41, written in the following form:

$$o_1 = \left(\frac{k_0}{m_0}\right) \cdot \left(\frac{l_1}{m_1}\right) \cdot \left(\frac{k_1}{m_1}\right) \cdot \cdots \cdot \left(\frac{l_{N-1}}{m_{N-1}}\right) \cdot \left(\frac{k_{N-1}}{m_{N-1}}\right) \cdot \left(\frac{l_N}{m_N}\right) t^K z\partial_z,$$

$$o_2 = \left(\frac{k'_0}{m'_0}\right) \cdot \left(\frac{l'_1}{m'_1}\right) \cdot \left(\frac{k'_1}{m'_1}\right) \cdot \cdots \cdot \left(\frac{l'_{N'-1}}{m'_{N'-1}}\right) \cdot \left(\frac{k'_{N'-1}}{m_{N'-1}}\right) \cdot \left(\frac{l'_{N'}}{m'_{N'}}\right) t^K w\partial_w.$$

The leading term of the obstruction vanishes when the sum of the pairings of a generator of the space H with these vectors is zero. This implies that

$$\begin{aligned} &\left(\tfrac{k_0}{m_0}\right) \cdot \left(\tfrac{l_1}{m_1}\right) \cdot \left(\tfrac{k_1}{m_1}\right) \cdot \cdots \cdot \left(\tfrac{l_{N-1}}{m_{N-1}}\right) \cdot \left(\tfrac{k_{N-1}}{m_{N-1}}\right) \cdot \left(\tfrac{l_N}{m_N}\right) \\ &= c \cdot \left(\tfrac{k'_0}{m'_0}\right) \cdot \left(\tfrac{l'_1}{m'_1}\right) \cdot \left(\tfrac{k'_1}{m'_1}\right) \cdot \cdots \cdot \left(\tfrac{l'_{N'-1}}{m'_{N'-1}}\right) \cdot \left(\tfrac{k'_{N'-1}}{m_{N'-1}}\right) \cdot \left(\tfrac{l'_{N'}}{m'_{N'}}\right) \end{aligned} \tag{6}$$

for some constant c. The constant c appears because the vectors $z\partial_z$ and $w\partial_w$ do not necessarily have the same value of the pairings with a generator of the space H.

So the equality above is equivalent to the statement that, when we consider the tropicalization with modulus τ of the pre-log curve $\varphi_0 : C_0 \to X_0$ (this is done by grafting the tropicalization of the pre-log curves with two components, corresponding to each neighboring two vertices, which, as noted above, can be considered as a tropicalization of a curve which intersects the toric divisors at four points.), then the

length of the paths $\mathcal{P}$ and $\mathcal{P}'$ satisfies

$$d_{(\Gamma,h)}(\mathcal{P}) = d_{(\Gamma,h)}(\mathcal{P}') + \frac{\log c}{\log \tau}. \tag{7}$$

Now we want to study the locus satisfying this condition in the space which parametrizes the genus one tropical curves of given degree. Although it is almost doubtless that such a space has a natural structure of a tropical variety, it does not seem to exist a result which is applicable to our situation (for genus zero case, see [1], for example). However, because of the genericity of the incidence conditions (see Subsection 6.3), we do not need general theory of moduli space of tropical curves.

According to Proposition 9, when we fix the combinatorial type, the space which parametrizes tropical curves of a given type is an unbounded open convex polyhedron (although it is stated for trivalent curves, it is easy to extend it so that it allows more general valency).

In this subsection, we are assuming that the tropical curve is immersive, in particular, trivalent. So when we fix the combinatorial type, the moduli space is a convex polyhedron, and the condition imposed by equation (7) cuts the polyhedron by a hyperplane. When τ is large enough, we can take this hyperplane arbitrarily close to the hyperplane given by

$$d_{(\Gamma,h)}(\mathcal{P}) = d_{(\Gamma,h)}(\mathcal{P}'), \tag{8}$$

which corresponds to the well-spacedness condition (precisely speaking, in a slightly extended sense, see Remark-Definition 61 below). As mentioned above, we will take the incidence conditions to be generic in the next section. So these conditions cut the polyhedron by generic piecewise linear conditions. In fact, assuming the incidence conditions to be generic, it suffices to consider only maximal dimensional faces of the polyhedron for enumeration problems. So, if τ is large enough, when we use the moduli spaces given by the equations (7) and (8), there is a canonical one-to-one correspondence between the solutions to the enumeration problem considered on these spaces.

On the other hand, when the tropical curve is immersive and the combinatorial type is fixed, the moduli space of pre-log curves of type (Γ, h) is locally isomorphic to the complexification of the moduli space of tropical curves containing (Γ, h). Namely, the moduli space of tropical curves is an open convex polyhedron which is contained in an affine subspace L defined over rational numbers in an affine space $\mathbb{Z}^m \otimes \mathbb{R}$ for some m. Consider $\mathbb{Z}^m \otimes \mathbb{C}^*$, and its universal cover $\mathbb{Z}^m \otimes \mathbb{C}$. Let $\bar{L}$ be the linear subspace parallel to L. Then $\bar{L} \otimes \mathbb{C}$ is a subspace of $\mathbb{Z}^m \otimes \mathbb{C}$, and since L is defined over rational numbers, its image in $\mathbb{Z}^m \otimes \mathbb{C}^*$ is a closed subspace of real dimension $2 \dim L$. So there is a well-defined 'complexification' of the moduli space of tropical curves, and this is locally isomorphic to the moduli space of corresponding pre-log curves

('locally' because there may be several pre-log curves corresponding to a fixed tropical curve).

Equation (7) or (8) for the lengths of the paths $\mathcal{P}, \mathcal{P}'$ cuts a hypersurface of the moduli space of tropical curves. On the other hand, Equation (6) cuts a complexified hypersurface of the moduli space of pre-log curves. Thus, the solution space of Equation (6) is also a complexification of the solution space of Equation (7) or (8).

Since for large τ, Equations (7) and (8) gives the same enumerative result, we can use Equation (8) (that is, the well-spacedness condition) for the enumeration problems.

Remark-Definition 61. From the point of view of parameterizing the smoothable tropical curves, it is convenient to extend the well-spacedness condition to the tropical curves defined over rational numbers (or real numbers). Clearly, this is done by replacing the integral distance by the path length $d_{(\Gamma,h)}$. From now on, we understand the well-spacedness condition in this extended sense.

Thus, the vanishing of the obstruction occurs, both macroscopically (meaning matching of the orders of t), and microscopically (meaning matching of the value of the residues), if and only if the well-spacedness condition is satisfied. We summarize it as follows, which is a strong form of Theorem 45.

Theorem 62. *Let (Γ, h) be an immersive genus one tropical curve. Then it is smoothable if and only if it satisfies the well-spacedness condition. Moreover, if $\varphi_0 : C_0 \to X_0$ is a pre-log curve of type (Γ, h) which is smoothable (see Definition 23), then the set of smoothable pre-log curves near φ_0 can be locally identified with the complexification of the moduli space of tropical curves near (Γ, h) which satisfies the well-spacedness condition.* □

6.2.4. **Non-immersive cases.** We briefly remark on the non-immersive case (we still assume Assumption C). There are two cases relevant to us, namely Example 1 and Example 2 (b) in Subsection 5.2. We discuss each of these cases separately.

We can assume that each four-valent vertex contributes to the leading term of the obstruction, since otherwise the set of such tropical curves is contained in a lower dimensional subset in the set of tropical curves satisfying the well-spacedness condition.

Case 1: Example 1. Macroscopically (i.e., seen as a subset of the polyhedral decomposition of $N_{\mathbb{R}}$ which determines the toric degeneration $\mathfrak{X}$), this case corresponds to a four-valent vertex such that one of the edges emanating from it has a weight of the form (w_1, w_2). In particular, it behaves just as a usual trivalent vertex in the image $h(\Gamma)$.

Microscopically, as we calculated in Subsection 5.2, the torically transverse curves corresponding to this type of four-valent vertices are

locally parametrized by $(a; b, c; a_1, \dots, a_{n-2}) \in \mathbb{C} \times (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{n-2}$, through the defining equations (we assume the smoothness of the corresponding toric surface, the general cases can be treated as in Example 1 in Subsection 5.2)

$$x^2 + ax + b + cz = 0, \quad y = 0, \quad w_1 = a_1, \dots, w_{n-2} = a_{n-2}.$$

There, we saw that the vanishing of the leading term of the obstruction was equivalent to the vanishing of the coefficient a (Proposition 46). Thus, such curves are parametrized by $(b, c; a_1, \dots, a_{n-2}) \in (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^{n-2}$. Then it is easy to see that, microscopically too, the tropicalization is a tropical curve whose image in $N_{\mathbb{R}}$ has just one trivalent vertex. That is, the tropicalizations (for any τ) of these curves are combinatorially the same as tropical lines, which is the same conclusion as the macroscopic consideration.

It is not difficult to extend the relation between the moduli of the pre-log curve and the length of the bounded edge of the tropicalization, as Remark 57, to this case. However, if a tropical curve with this type of vertex satisfies a generic incidence condition, then the leading term of the obstruction contributed from this vertex must vanish, so the length of the path from it to the loop does not matter. This is the same for the next case, too.

In this case, the moduli space of these tropical curves is a facet of an open convex polyhedron which is the moduli space of trivalent tropical curves obtained by deforming the four-valent vertex into two trivalent vertices. But, the fact that the moduli space of corresponding pre-log curves with vanishing obstruction is obtained as the complexification is the same as in the immersive case. Thus, in this case, we reach to the same conclusion as Theorem 62 (with extended well-spacedness condition, Definition 51).

Case 2: Example 2 (b). Macroscopically, this case corresponds to a four-valent vertex such that the image of the corresponding vertex in $h(\Gamma)$ also has four different edges (in other words, the weight (see Definition 14 (iii)) of every edge is given by a single integer).

Microscopically, as we calculated in Subsection 5.2, the torically transverse curves corresponding to this type of four-valent vertices are parametrized by $(a, b, e, f; a_2, \dots, a_{n-2}) \in (\mathbb{C}^*)^4 \times (\mathbb{C}^*)^{n-3}$, through the defining equations

$$ax + z + b = 0, \quad ex + w_1 + f = 0, \quad y = 0, \quad w_2 = a_2, \dots, w_{n-2} = a_{n-2}.$$

Again we assumed the smoothness for simplicity.

According to the calculation there, one sees that the vanishing of the leading term of the obstruction is equivalent to

$$\frac{a}{b} = c \cdot \frac{e}{f}$$

for some fixed constant c. One sees that, this condition implies that the tropicalization of the curve as above is a tropical curve with only one vertex which is four-valent, in the limit $\tau \to \infty$.

For large but finite τ, the tropicalization has two trivalent vertices. However, since the length of the bounded edge is fixed (explicitly, $\frac{\log c}{\log \tau}$), the moduli is isomorphic to $\mathbb{R}^3$, determined by the place of one of the vertices. Thus, in this case too, the moduli space of pre-log curves with vanishing obstruction is isomorphic to the complexification of the moduli of the tropical curves with the condition for the length of the edges. Also, by the same reasoning as in the immersive case, for large enough τ, we can replace the condition for the length by the well-spacedness condition for enumeration problems.

Thus, we obtain the following generalization of Theorem 62.

Theorem 63. *The same conclusion as Theorem 62 holds for not necessarily immersive genus one superabundant tropical curves satisfying Assumption C.* □

6.3. **Enumerative result.** In this subsection, using the Kuranishi map and the results of [9], we deduce the enumerative correspondence between tropical curves and holomorphic curves for genus one case, analogous to Theorem 8.3 of [9].

First we briefly recall some terminologies concerning incidence conditions from [9], which are necessary to state the result. See [9] for more details.

Definition 64. For $\mathbf{d} = (d_1, \dots, d_l) \in \mathbb{N}^l$, an *affine constraint* of codimension $\mathbf{d}$ is an l-tuple $\mathbf{A} = (A_1, \dots, A_l)$ of affine subspaces $A_i \subset N_{\mathbb{R}}$, defined over rational numbers, with

$$\dim A_i = n - d_i - 1.$$

An l-marked tropical curve $(\Gamma, \mathbf{E}, h)$ *matches* the affine constraint $\mathbf{A}$ if

$$h(E_i) \cap A_i \neq \emptyset, \quad i = 1, \dots, l.$$

Let us fix a degree $\Delta : N \setminus \{0\} \to \mathbb{N}$. Now let

$$\mathbf{L} = (L_1, \dots, L_l)$$

be a set of linear subspaces of $N_{\mathbb{Q}}$, with $\operatorname{codim} L_i = d_i + 1$. Assume

$$\sum_{i=1}^{l} d_i = \dim E$$

(the space E is defined in Proposition 55).

Let

$$\mathbf{A} = (A_1, \dots, A_l),$$

A_i is parallel to L_i, be a general affine constraint (see Definition 2.3, [9], there it is defined for rational tropical curves, but it obviously

generalizes to any genus) for tropical curves of the same combinatorial type as (Γ, h).

Let $\mathscr{P}$ be a polyhedral decomposition of $N_{\mathbb{R}}$ containing $h(\Gamma)$ in its 1-skeleton. As in [9], we assume that any intersection point of A_i with $h(\Gamma)$ is a vertex of $\mathscr{P}$. For each face $\Xi \in \mathscr{P}$, let $C(\Xi)$ be the closure of the cone spanned by $\Xi \times \{1\}$ in $N_{\mathbb{R}} \times \mathbb{R}$:

$$C(\Xi) = \overline{\{a \cdot (n, 1) \mid a \geq 0, n \in \Xi\}}.$$

Then

$$\widetilde{\Sigma}_{\mathscr{P}} = \{\sigma \subset C(\Xi) \mid \sigma \text{ is a face of } C(\Xi), \Xi \in \mathscr{P}\}$$

is a fan covering $N_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$. Lemma 3.3 of [9] shows that, if we identify $N_{\mathbb{R}}$ with $N_{\mathbb{R}} \times \{0\} \subset N_{\mathbb{R}} \times \mathbb{R}$, then

$$\Sigma_{\mathscr{P}} = \{\sigma \cap (N_{\mathbb{R}} \times \{0\}) \mid \sigma \in \widetilde{\Sigma}_{\mathscr{P}}\}$$

is a complete fan in $N_{\mathbb{R}}$ and defines a toric variety associated to (Γ, h). The fan $\widetilde{\Sigma}_{\mathscr{P}}$ defines a toric degeneration $\mathfrak{X} \to \mathbb{C}$ of X defined respecting (Γ, h).

For an affine subspace A_i of $N_{\mathbb{R}}$, let $C(A_i)$ be the cone

$$C(A_i) = \overline{\{a \cdot (n, 1) \mid a \geq 0, n \in A_i\}} \subset N_{\mathbb{R}} \times \mathbb{R},$$

and let $LC(A_i)$ be the linear subspace of $N_{\mathbb{R}} \times \mathbb{R}$ spanned by $C(A_i)$. We write by $\mathbb{G}(LC(A_i))$ the subtorus of the big torus $\mathbb{C}^* \otimes (N \oplus \mathbb{Z})$ acting on $\mathfrak{X}$.

Let

$$P_1, \dots, P_l \in X(\Sigma)$$

be general points. Let

$$Z_i = \overline{\mathbb{G}(LC(A_i)).P_i}$$

be the closure of the orbit through P_i. We write by

$$\mathbf{Z} = (Z_1, \dots, Z_l)$$

the incidence conditions for holomorphic curves.

Knowing the correspondence between well-spaced tropical curves and smoothable pre-log curves (Theorems 62, 63), the number of different families of smoothings incident to the subvarieties $\mathbf{Z} = \{Z_i\}$ is calculated by the same method as [9]. Namely,

(i) Count the number of stable maps to X_0 satisfying the incidence conditions $\{Z_i \cap X_0\}$ (Proposition 5.7, [9]).
(ii) For each stable map above, calculate the number of different families of smoothings of it (Propositions 7.1, 7.3 and Section 8 of [9]).

Fix a marking $\mathbf{E}$ of Γ. Using the same notation as [9], let $\mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A})$ be the number of *unmarked* pre-log curves $\varphi_0 : C_0 \to X_0$ of type (Γ, h) satisfying the incidence conditions. Let $X_{0,i}$ be the component of X_0

corresponding to the vertex $\mathfrak{V}_i \in A_i \cap h(\Gamma)$, and $C_{0,i}$ be the component of C_0 mapped to $X_{0,i}$.

We remark that if (Γ, h) is not immersive, so there is a four-valent vertex $\mathfrak{V}$ in $h(\Gamma)$, and if v_1, v_2 are the vertices of Γ mapped to $\mathfrak{V}$, the node between the components in C_0 corresponding to v_1 and v_2 are smoothed, and these components are merged into one rational component.

Also note that as in [9], we add divalent vertices to Γ at the intersection points of the constraints A_i with the images of the marked edges $\mathfrak{E}_i = h(E_i)$, and correspondingly we add rational components to C_0.

An exceptional case is, when an edge $\mathfrak{E}$ with weight (w_1, w_2) intersects a constraint A. In this case, if $E_1, E_2 \subset \Gamma$ are inverse images of $\mathfrak{E}$ and at least one of them is marked, then we add a divalent vertex to each E_i even if one of them is not marked. This is just for the well-definedness of the map from C_0 to X_0. Theoretically, there is a possibility that both E_1 and E_2 are marked, but this does not happen when the constraints are generic.

Let $\delta_i(\Gamma, \mathbf{E}, h, \mathbf{A})$ be the number of intersection points of Z_i and $\varphi_0(C_{0,i})$ (see Remark 67). This number can be calculated from the data of (Γ, h) and A_i (see Definition 66 below and Remark 5.8, [9]). Then

$$\widetilde{\mathfrak{D}}(\Gamma, \mathbf{E}, h, \mathbf{A}) := \mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A}) \cdot \prod_i \delta_i(\Gamma, \mathbf{E}, h, \mathbf{A})$$

is the number of *marked* pre-log curves of type (Γ, h) satisfying the incidence conditions.

Then, the argument in Section 7 of [9] shows that given a marked pre-log curve

$$\varphi_0 : (C_0, \mathbf{x}) \to (X_0, \mathbf{Z}),$$

it has $w(\Gamma, \mathbf{E}, h)$ (Definition 16) different families of smoothings.

Remark 65. *Note that this argument is valid even if some of the edges have weight of the form $(w_1, \dots, w_k)$ (in our genus one case, generically only $k = 2$ case can appear on the unbounded edges).*

So it suffices to calculate $\mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A})$ in our situation. In Proposition 5.7, [9], it is defined as the index of the inclusion of the lattices,

(9)
$$\begin{array}{rcl} Map(\Gamma^{[0]}, N) & \to & \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty} N/\mathbb{Z}u_{(\partial^- E, E)} \times \prod_{i=1}^l N/(\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N, \\ h & \mapsto & ((h(\partial^+ E) - h(\partial^- E))_E, (h(\partial^- E_i))_i). \end{array}$$

Here $\partial^\pm : \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty \to \Gamma^{[0]}$ is an arbitrary chosen orientation of the bounded edges, that is, $\partial E = \{\partial^- E, \partial^+ E\}$. For $E \in \Gamma^{[1]}_\infty$, $\partial^- E$ denotes the unique vertex adjacent to E.

Definition 66. Using this notation, the number of intersections $\delta_i(\Gamma, \mathbf{E}, h, \mathbf{A})$ is given by the product

$$\delta_i(\Gamma, \mathbf{E}, h, \mathbf{A}) = w(E_i) \cdot [(\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N : \mathbb{Z}u_{(\partial^- E_i, E_i)} + L(A_i) \cap N].$$

Remark 67. *Note that the actual number of intersection with Z_i and $\varphi_0(C_0)$ is in general larger than δ_i, when the image $h(E_i)$ has weight of the form $(w_1, \dots, w_k)$. In fact, this number is*

$$w_s(h(E_i)) \cdot [(\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N : \mathbb{Z}u_{(\partial^- E_i, E_i)} + L(A_i) \cap N],$$

here $w_s(h(E_i))$ is the total additive weight (Definition 14). However, only $w(E_i) \cdot [(\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N : \mathbb{Z}u_{(\partial^- E_i, E_i)} + L(A_i) \cap N]$ of them are compatible with the marking.

6.3.1. *Calculation of index I. Immersive cases.* In our case with obstruction, the rank of the module on the left hand side of Equation (9) is larger than that of the right hand side. We show how to modify it in our case, first assuming (Γ, h) is immersive. Non-immersive cases are treated later.

So assuming (Γ, h) is immersive, we first consider the part of the above map

$$\begin{array}{rcl} P_1 : Map(\Gamma^{[0]}, N) & \to & \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty} N / \mathbb{Z}u_{(\partial^- E, E)}, \\ h & \mapsto & (h(\partial^+ E) - h(\partial^- E))_E, \end{array}$$

and let us denote by $\mathfrak{N}$ its kernel. The space $\mathfrak{N}$ (more precisely, when tensored with $\mathbb{R}$) contains the moduli space of tropical curves of the same combinatorial type as (Γ, h) as a maximal dimensional convex polyhedron.

Let a be the dimension of the dual obstruction space H. If v is a vertex of Γ and $\mathcal{P}_v$ is the unique path from v to the loop, then the path length $d_{(\Gamma,h)}(\mathcal{P}_v)$ is an affine function on $\mathfrak{N}$.

Let $\mathcal{M} \subset \mathfrak{N}$ be the set of tropical curves of the same combinatorial type as (Γ, h) satisfying the well-spacedness condition. The set $\mathcal{M}$ is defined as follows. Namely, let

$$\{\mathfrak{a}_1, \dots, \mathfrak{a}_a\}$$

be a basis of H. Let

$$v_{i,1}, \dots, v_{i,k_i}$$

be vertices of $h(\Gamma)$ satisfying the following condition. That is, let $L_{i,j}$ be the subspace of $N_\mathbb{R}$ spanned by the edges emanating from $v_{i,j}$ (this is two dimensional because we are assuming (Γ, h) is immersive). Then the condition is that there is a vector in $L_{i,j}$ such that the pairing with $\mathfrak{a}_i$ is not zero. Let $d_{i,j}$ be the length of the path from the vertex $v_{i,j}$ to the loop. As noted above, this is an affine function on $\mathfrak{N}$. In particular, $d_{i,j}$ can be considered as a monomial on $\mathfrak{N}$ in the tropical sense. Then the set $\mathcal{M}$ is defined by the tropical polynomials

$$\max\{-d_{i,1}, \cdots, -d_{i,k_i}\}, \quad i = 1, \dots, a.$$

In particular, $\mathcal{M}$ has a natural structure of tropical manifold.

Let $X \in \mathcal{M}$ be a point corresponding to a generic well-spaced tropical curve (Γ, h_X). Here generic means that if $\{f_1, \dots, f_a\}$ is the set of

tropical polynomials defining $\mathcal{M}$ near X, then for each f_i, just two of the terms of it take the maximum.

Then, for each i, two vertices $v_{i,1}, v_{i,2}$ of Γ, which give the maximum of f_i, are determined. Let $U_{X,\mathbb{R}}$ be a suitable neighborhood of X in $\mathfrak{N} \otimes \mathbb{R}$. If necessary, we scale (Γ, h_X) and the incidence conditions so that there are enough number of integral points in $U_{X,\mathbb{R}}$. Let U_X be the set of integral points in $U_{X,\mathbb{R}}$, that is, $U_X = U_{X,\mathbb{R}} \cap \mathfrak{N}$. On U_X, we define the map

$$P_2 : U_X \to \left(\prod_{i=1}^{l} N/\left((\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N\right)\right) \times \mathbb{Z}^a$$

by

$$Y \mapsto \left((h_Y(\partial^- E_i))_i, (d_{j,1} - d_{j,2})_{j=1,\dots,a}\right).$$

From P_1 and P_2, an inclusion of lattices

$$Map(\Gamma^{[0]}, N) \to \prod_{E \in \Gamma^{[1]} \setminus \Gamma^{[1]}_\infty} N/\mathbb{Z}u_{(\partial^- E, E)} \times \left(\prod_{i=1}^{l} N/\left((\mathbb{Q}u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N\right)\right) \times \mathbb{Z}^a$$

is defined. Note that this may depend on the choice of a point X of $\mathcal{M}$.

Definition 68. For each $Y \in P_2^{-1}(0)$, we define the number $\mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A}, Y)$ to be the lattice index of the above inclusion of lattices. Furthermore, we define

$$\widetilde{\mathfrak{D}}(\Gamma, \mathbf{E}, h, \mathbf{A}, Y) := \mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A}, Y) \cdot \prod_{i=1}^{l} \delta_i(\Gamma, \mathbf{E}, h, \mathbf{A})$$

6.3.2. *Calculation of index II. General cases.* In general, some vertices of $h(\Gamma)$ may be four-valent. These tropical curves are parametrized by appropriate face (more precisely, the interior of such a face) of the closure of the convex polyhedron in $\mathfrak{N}$ appeared in the immersive cases. Instead, we can also think that they are immersive tropical curves, with the domain curve Γ modified, namely, some of the edges of Γ are contracted or merged. We take the latter viewpoint and fix a combinatorial type $(\Gamma_1, u : F\Gamma_1 \to N)$ (Definition 6). Let $v_1, \dots, v_b$ be the four-valent vertices of Γ_1. As in Subsection 6.2.4, we assume that each four-valent vertex contributes to the leading term of the obstruction.

Then we consider a map as above:

$$\begin{aligned} P_1 : Map(\Gamma_1^{[0]}, N) &\to \textstyle\prod_{E \in \Gamma_1^{[1]} \setminus (\Gamma_1)^{[1]}_\infty} N/\mathbb{Z}u_{(\partial^- E, E)}, \\ h &\mapsto (h(\partial^+ E) - h(\partial^- E))_E, \end{aligned}$$

and let $\mathfrak{N}_1$ be its kernel. Let $\mathcal{M}_1 \subset \mathfrak{N}_1$ be the set of tropical curves of given combinatorial type satisfying the well-spacedness condition, and let $X = (\Gamma_1, h_X) \in \mathcal{M}_1$ be an element. Then, (Γ_1, h_X) satisfies the

following condition:

For each v_i, let $\mathfrak{L}_i$ be the linear subspace of $N_{\mathbb{R}}$ spanned by the edges of $h_X(\Gamma_1)$ emanating from $h_X(v_i)$. Let $\mathfrak{L}$ be the subspace of $N_{\mathbb{R}}$ spanned by $\mathfrak{L}_1, \dots, \mathfrak{L}_b$ and $\bar{A}$ (the plane spanned by the loop). Then, $\mathfrak{L}$ has codimension $a - b$ in $N_{\mathbb{R}}$.

Let

$$\{\mathfrak{a}_1, \dots, \mathfrak{a}_{a-b}\}$$

be a basis of $\mathfrak{L}^{\perp} \subset H$. Using this basis, we can construct a set of tropical polynomials $\{f_1, \dots, f_{a-b}\}$ on $\mathfrak{N}_1$ as in the immersive case. Then, the set $\mathcal{M}_1$ is a subset of the tropical variety V defined by these tropical polynomials. Note that, contrary to the immersive case, some of the points of V may not be contained in $\mathcal{M}_1$, because some trivalent vertices of $h(\Gamma_1)$ with the property that the leading terms of the contributions to the obstruction from them have the directions in the space spanned by $\mathfrak{L}_1 \cup \cdots \cup \mathfrak{L}_b$, can be closer to the loop than four-valent vertices.

Now suppose $X \in \mathcal{M}_1$ is generic in the same sense as before. Then, as in the immersive case, we define the map

$$P_2 : U_X \to \left(\prod_{i=1}^{l} N / \left((\mathbb{Q} u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N \right) \right) \times \mathbb{Z}^{a-b},$$

where U_X is a suitable neighborhood of X in $\mathfrak{N}_1$. From P_1 and P_2, an inclusion of lattices

$$Map(\Gamma_1^{[0]}, N) \to \prod_{E \in \Gamma_1^{[1]} \setminus (\Gamma_1)_{\infty}^{[1]}} N/\mathbb{Z} u_{(\partial^- E, E)} \times \left(\prod_{i=1}^{l} N / \left((\mathbb{Q} u_{(\partial^- E_i, E_i)} + L(A_i)) \cap N \right) \right) \times \mathbb{Z}^{a-b}$$

is defined. Then the index is defined as before.

Definition 69. We define the number $\mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A}, X)$ to be the lattice index of the above inclusion of lattices. Furthermore, we define

$$\widetilde{\mathfrak{D}}(\Gamma, \mathbf{E}, h, \mathbf{A}, X) := \mathfrak{D}(\Gamma, \mathbf{E}, h, \mathbf{A}, X) \cdot \prod_{i=1}^{l} \delta_i(\Gamma, \mathbf{E}, h, \mathbf{A})$$

6.3.3. *Enumerative invariants.* The numbers $N_{1,\Delta}^{\mathrm{trop}}(\mathbf{A})$ and $N_{1,\Delta}^{\mathrm{alg}}(\mathbf{L})$ are defined in the same way as in Definitions 8.1 and 8.2 of [9].

Definition 70. The number $N_{1,\Delta}^{\mathrm{trop}}(\mathbf{A})$ is the weighted count of the genus one tropical curves of degree Δ matching the affine constraint $\mathbf{A}$ and satisfying the well-spacedness condition:

$$N_{1,\Delta}^{\mathrm{trop}}(\mathbf{A}) = \sum_{(\Gamma, \mathbf{E}, h_Y) \in \mathfrak{T}_{1,l,\Delta}^{ws}(\mathbf{A})} w(\Gamma, \mathbf{E}, h_Y) \cdot \widetilde{\mathfrak{D}}(\Gamma, \mathbf{E}, h, \mathbf{A}, Y).$$

Here $\mathfrak{T}^{ws}_{1,l,\Delta}(\mathbf{A})$ is the set of well-spaced genus one l-marked tropical curves of degree Δ matching $\mathbf{A}$.

Note that non-superabundant curves are automatically well-spaced.

On the other hand, $N^{\mathrm{alg}}_{1,\Delta}(\mathbf{L})$ is the genuine count of the genus one stable maps of degree Δ (see Section 8 of [9]), satisfying the incidence condition $\mathbf{Z}$ (as in [9], we write $N^{alg}_{1,\Delta}(\mathbf{L})$ instead of $N^{alg}_{1,\Delta}(\mathbf{Z})$ because this number depends only on $\mathbf{L}$, not on $\mathbf{Z}$, if $\mathbf{Z}$ is generally chosen).

Then using Proposition 55, one can prove the analogue of Proposition 7.3 of [9], with $H^0(\mathcal{N}_{C_0/X_0})$ replaced by the subspace E, which is the local model of the moduli space $\mathcal{M}$. Then the proof of Theorem 8.3 of [9] applies and we have the following enumerative result.

Theorem 71. *The numbers $N^{\mathrm{trop}}_{1,\Delta}(\mathbf{A})$ and $N^{\mathrm{alg}}_{1,\Delta}(\mathbf{L})$ are finite, do not depend on the incidence conditions as long as they are generically chosen, and the equality*

$$N^{\mathrm{trop}}_{1,\Delta}(\mathbf{A}) = N^{\mathrm{alg}}_{1,\Delta}(\mathbf{L})$$

holds. □

We note that when the affine constraint $\mathbf{A}$ is generic, then any tropical curve matching $\mathbf{A}$ satisfies Assumption C, according to Lemma 48.

7. Correspondence theorem for superabundant curves III: Higher genus

In this section, we study the smoothability of the pre-log curves of type (Γ, h), where (Γ, h) is a superabundant genus g tropical curve. *We still assume Assumption A.* As we saw in Remark 33 and Example 50, this is too strong an assumption for the enumeration problems. So the results for higher genus curves are necessarily weaker than those for genus one case. In particular, we do not pursue enumeration problem for higher genus curves in this paper. Instead, in this section we establish a general framework and leave these problems to future study.

7.1. **Supports of dual obstruction vectors.**

We know the description of the dual space H of the obstructions by Theorem 30. There, each bouquet (Definition 12) of Γ was decomposed into piecewise linear segments $\{l_m\}$. An element $\mathfrak{a} \in H$ associates a vector $u_m \in (N_{\mathbb{C}})^{\vee}$ to each l_m.

Definition 72. The *support* of $\mathfrak{a} \in H$ is the union of those segments $\{l_m\}$ such that $u_m \neq 0$. We write it as $Supp(\mathfrak{a})$.

It is clear that for any $\mathfrak{a}$, $Supp(\mathfrak{a})$ is a union of loops (in particular, there is no one valent end).

Now we decompose Γ as in Figure 2 into the loop part and the other part. Let $L = \cup L_i$ be the loop part where each L_i is a bouquet. We

can take a basis of H so that for each element $\mathfrak{a}$ of it, there is some i such that $Supp(\mathfrak{a}) \subset L_i$. We write the part of this basis with this property as

$$\{\mathfrak{a}_{i,1}, \dots, \mathfrak{a}_{i,j_i}\}.$$

Note that since we keep Assumption A, we can identify L_i with its image by h.

The basic strategy for the study of the smoothability of pre-log curves is the same as in the genus one case. Namely, starting from a pre-log curve $\varphi : C_0 \to \mathfrak{X}$ of type (Γ, h), we proceed as follows:

(1) Assume we have constructed a k-th order smoothing $\varphi_k : C_k \to \mathfrak{X}$ of φ_0. For each smoothing corresponding to a section $\mathfrak{n} \in H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ near φ_k, we calculate the obstruction.
(2) From these obstructions, define the Kuranishi map.
(3) Study the zero locus of the Kuranishi map.

7.2. **Kuranishi map.** In this subsection, we define the Kuranishi map for (lifts of) pre-log curves $\varphi_0 : C_0 \to \mathfrak{X}$ of type (Γ, h). There is one difference from the genus one case concerning the process (1) above. Namely, the Kuranishi map may be defined only on some subset of $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$. Now we explain this point.

In genus one case, the domain of the Kuranishi map was $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$. This was possible because the components of $C_0 \setminus C_L$ were rational curves and so smoothable in any order of t. In particular, given a k-th order smoothing $\varphi_k : C_k \to \mathfrak{X}$, any element of $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ gives smoothings of these components, and it enables us to calculate the obstructions, and the Kuranishi map.

On the other hand, for higher genus case, fixing a connected component L_i of loops of Γ, the components of $C_0 \setminus C_{L_i}$ need not be always smoothable. Thus, in the situation above, some elements of $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ may not correspond to smoothings of these components, and for them we cannot define obstructions for the smoothing to the $(k+1)$-th order. As a result, we cannot define the Kuranishi map on these elements.

Now assume we have a k-th order smoothing $\varphi_k : C_k \to \mathfrak{X}$. Let $\mathcal{B}_{L_i}$ be the set of connected components of $C_0 \setminus C_{L_i}$. Note that by construction, the closure of each component $B \in \mathcal{B}_{L_i}$ has just one intersection with C_{L_i}.

Definition 73. Let W_{φ_k, L_i} be the subset of $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ defined by the following property. Namely, $\mathfrak{n} \in H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ belongs to W_{φ_k, L_i} if and only if for each $B \in \mathcal{B}_{L_i}$, there is the k-th order smoothing of $\varphi_0|_B$ obtained by perturbing φ_k by $\mathfrak{n}|_B$.

For such $\mathfrak{n}$, we can calculate the obstruction at L_i. Namely, for each $B \in \mathcal{B}_{L_i}$, we calculate it just as Step 6 in Section 5 (Definition 43).

Definition 74. We write by $o(\varphi_k; \mathfrak{n}; L_i, B)$ the obstruction calculated in this way for $B \in \mathcal{B}_{L_i}$. Then, we define the obstruction for L_i by

$$o(\varphi_k; \mathfrak{n}; L_i) = \sum_{B \in \mathcal{B}_{L_i}} o(\varphi_k; \mathfrak{n}; L_i, B).$$

This is defined only for $\mathfrak{n} \in W_{\varphi_k, L_i}$.

Definition 75. Let us define the subset W_{φ_k} of $H^0(\mathcal{N}_{C_0/X_0}) \otimes \mathbb{C}[t]/t^k$ by

$$W_{\varphi_k} = \bigcap_{L_i} W_{\varphi_k, L_i}.$$

Definition 76. For each L_i, we define the *Kuranishi map* of order k at φ_k and L_i

$$\mathcal{K}_{L_i} : W_{\varphi_k, L_i} \to (\mathbb{C}[t]/t^{k+1})^{j_i}$$

by

$$\mathfrak{n} \mapsto \{\langle \mathfrak{a}_{i,m}, o(\varphi_k; \mathfrak{n}; L_i) \rangle\}_{m=1,\dots,j_i}.$$

Here $\langle \cdot , \cdot \rangle$ is the natural pairing between the germs of $N_{\mathbb{C}}$-valued rational sections and elements of H (Definition 53).

The Kuranishi map of order k at φ_k is the collection of $\mathcal{K}_{L_i}$:

$$\begin{aligned}&\mathcal{K} : W_{\varphi_k} \to (\mathbb{C}[t]/t^{k+1})^a,\\ &\mathfrak{n} \mapsto \{(\mathcal{K}_{L_i}(\mathfrak{n}))\}_{L_i} = \{\langle \mathfrak{a}_{i,m}, o(\varphi_k; \mathfrak{n}; L_i) \rangle\}_{L_i;\, m=1,\dots,j_i}.\end{aligned}$$

Here $a = \dim H$ as usual.

By construction, the following holds.

Proposition 77. *A perturbation of φ_k corresponding to $\mathfrak{n} \in W_{\varphi_k}$ can be lifted to a $(k+1)$-th order smoothing if and only if $\mathcal{K}(\mathfrak{n}) = 0$.* □

Remark 78. *Strictly speaking, in the statement of this proposition, the phrase 'a perturbation of φ_k corresponding to $\mathfrak{n} \in W_{\varphi_k}$' is somewhat imprecise, because not every element of W_{φ_k} corresponds to a k-th order lift of φ_0. That is, the restriction of φ_0 to each component of $\mathcal{B}_{L_i}$ for each L_i has a lift corresponding to $\mathfrak{n}$, but φ_0 itself does not necessarily have a lift corresponding to $\mathfrak{n}$. This is the same for the genus one case, too.*

As one may anticipate, it is very difficult to calculate the Kuranishi map in general situation. However, in low genus it is quite manageable. Also, in several situations general results can be proved. The rest of this section is devoted to give some of such examples.

7.3. **Genus two example.** As noted above, for low genus curves we can study Kuranishi map in many cases. We leave the detailed study to future work and here we give some simple examples.

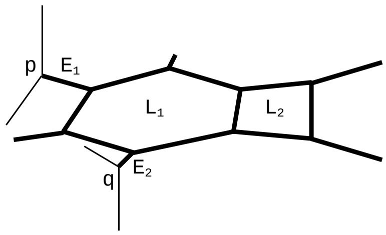

FIGURE 19. The edges drawn by bold lines are contained in a fixed affine plane.

Example 79. Consider a pre-log curve associated to the following immersive genus two superabundant tropical curve (Γ, h) (Figure 19) in $\mathbb{R}^3$:

There is the unique connected component of the loops: $L = L_1 \cup L_2$. For an element $\mathfrak{a}$ of H, there are three possibilities for the support $Supp(\mathfrak{a})$: L_1, L_2 or $L_1 \cup L_2$.

As a basis of H, we can take

$$\mathfrak{a}_1, \mathfrak{a}_2 \in H$$

such that

$$Supp(\mathfrak{a}_1) = L_1, \ \ Supp(\mathfrak{a}_2) = L_2.$$

In this case, we can calculate the obstruction by a straightforward extension of the calculation in Section 5. Since $\mathfrak{a}_2$ couples with these obstructions always trivially, the necessary and sufficient condition for the existence of a smoothable pre-log curve is that, there is a (lift of) pre-log curve whose obstruction couples with $\mathfrak{a}_1$ trivially, too. As in genus one case, one sees that this is equivalent to the condition that the integral lengths of the edges E_1 and E_2 are the same when they have weight one. The case with higher weight is similar.

In fact, in this case the zero locus of the Kuranishi map can be studied exactly as in the genus one case, and the moduli space of smoothable pre-log curves is locally isomorphic to the complexification of the moduli space of tropical curves satisfying the (straightforwardly extended) well-spacedness condition, see Subsection 7.6. In particular, the dimension of the moduli space is larger than expected, whenever it is non-empty.

On the other hand, if there is another source of obstruction as in Figure 20, then there is no smoothable pre-log curve of the corresponding type whatever the length of F_1 is.

7.4. **The case $\mathcal{K} = 0$.** For some tropical curves (Γ, h), the Kuranishi map becomes zero, so that any pre-log curve of type (Γ, h) is smoothable. We use the same notations as in the previous subsection. Take a bouquet L_i of Γ. We cut L_i at the trivalent vertices as in Section 4 and let $\{l_{L_i,m}\}$ be the set of connected components obtained by this

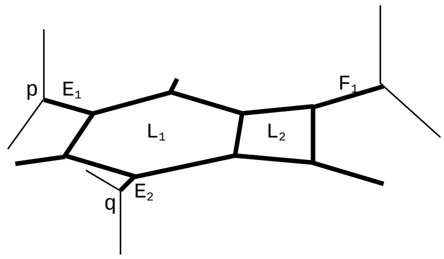

FIGURE 20.

process. Let $U_{L_i,m}$ be the subspace of $N_{\mathbb{R}}$ spanned by the direction vectors of the edges of $l_{L_i,m}$, as in Section 4.

For any $B \in \mathcal{B}_{L_i}$, we choose l_{L_i,m_B} so that it is the unique element of $\{l_{L_i,m}\}$ intersecting the closure of B. Let V_B be the subspace of $N_{\mathbb{R}}$ spanned by the direction vectors of the edges of B. The following is obvious from the calculation of the obstructions in Section 5, because the pairing of the residues with elements of H are zero when the condition in this proposition is satisfied.

Proposition 80. *Suppose that a tropical curve* (Γ, h) *(we assume Assumption A) satisfies the following condition: For any bouquet* L_i *and* $B \in \mathcal{B}_{L_i}$*, the inclusion*

$$V_B \subset U_{L,m_B}$$

holds. Then, the Kuranishi map is zero at any order and at any (lifts of) pre-log curve of type (Γ, h). $\square$

Corollary 81. *Suppose a tropical curve* (Γ, h) *has a unique bouquet* L*, and all the components of the complement* $h(\Gamma) \setminus h(L)$ *are unbounded edges. Then the Kuranishi map is zero at any order and at any (lifts of) pre-log curve of type* (Γ, h). $\square$

In particular, given any finite, weighted, trivalent, closed (that is, there is no open edge) graph G, and a piecewise linear embedding g of it to some $\mathbb{R}^n$ with the condition that it satisfies the balancing condition (Definition 3) at the trivalent vertices, then there is a smoothable tropical curve $(\Gamma, h : \Gamma \to \mathbb{R}^n)$ which is an extension of $g : G \to \mathbb{R}^n$ in the sense that there is an inclusion $\iota : G \to \Gamma$ such that $h \circ \iota = g$. Such a (Γ, h) is obtained just by adding an unbounded edge to each di-valent vertex of the image $g(G)$ so that the resulting trivalent vertex satisfies the balancing condition.

We give a simple example to which this proposition applies.

Example 82. We consider the following immersed genus two tropical curve in $\mathbb{R}^3$ (Figure 21).

This is a modification of the tropical curve Γ_1 in Subsection 5.2.1. The modification is done to each unbounded edge, so that the direction

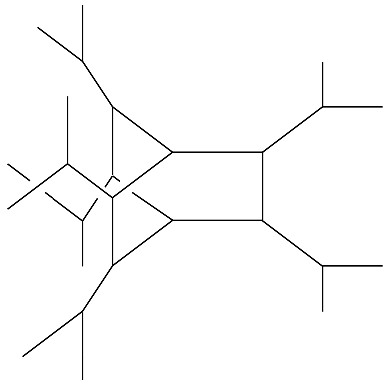

FIGURE 21.

vectors of the unbounded edges become

$$(0, 0, \pm 1), \quad (1, 0, 0), \quad (0, 1, 0), \quad (-1, -1, 0).$$

We saw that this is a superabundant curve and $\dim H = 1$. Moreover, the support of a nonzero element of H is the whole loop part (Figure 22), and it is decomposed into the union of edges as in Figure 4.

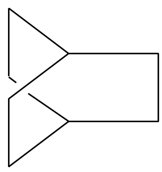

FIGURE 22.

It is easy to see that this example satisfies the assumption of Proposition 80, so any pre-log curve of type (Γ, h) is smoothable.

One calculates that the expected dimension of this curve is 12. On the other hand, since $\dim H = 1$, the moduli space of the tropical curve has dimension 13. Since any pre-log curve of type (Γ, h) is smoothable, the moduli space of the holomorphic curves obtained by smoothing these pre-log curves also has dimension 13. In fact, this corresponds to a genus two curve in $\mathbb{P}^2 \times \mathbb{P}^1$ which is contained in a subvariety $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^2 \times \mathbb{P}^1$. In $\mathbb{P}^1 \times \mathbb{P}^1$, this curve has degree $(3, 2)$ and one sees that the moduli space of those curves has dimension 11 (this moduli space is smooth of expected dimension). There is a two dimensional freedom to move $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^1$, and the curves have total 13 dimensional moduli parameter.

It is possible to put 13 dimensional incidence conditions to tropical/holomorphic curves of type (Γ, h), and count curves satisfying these conditions. However, in this case there is no reason to expect that the counting number is independent of the position of the incidence conditions (compare with Theorem 71).

7.5. **Realizability of tropical curves.** Roughly speaking, any finite graph can be realized as a tropical curve which is smoothable. To state the result more precisely, we introduce the following notion.

Definition 83. Let Γ be an abstract finite graph without a one-valent vertex, possibly with non-compact edges. We say that another graph Γ' is a *stabilization* of Γ if Γ' contains Γ as a subgraph and each component of $\Gamma' \setminus \Gamma$ is a tree, which is attached to Γ at just one point.

In other words, a stabilization Γ' is a graph obtained from Γ by attaching some trees.

Let Γ be any trivalent graph as above. We consider an embedding h of Γ to $\mathbb{R}^n$ for some n, so that the resulting pair (Γ, h) is a tropical curve (with suitable weights on the edges of Γ) and moreover smoothable. It is easy to see that not every curve Γ has an embedding as a tropical curve. However, it is also easy to see that there is a stabilization Γ' of Γ such that Γ' has an embedding as a tropical curve.

According to [4], any immersed tropical curve in $\mathbb{R}^2$ is smoothable (at least when the vertices are at most four-valent and each four-valent vertex is a normal crossing of two edges). So we might say that if we allow self-intersections to the map h, then any trivalent graph has a stabilization Γ' which is also trivalent, such that there is a map $h : \Gamma' \to \mathbb{R}^2$ and (Γ', h) is a smoothable tropical curve.

But when we require h to be embedding (there are some situations in which this is desirable. For example, when Γ has a ribbon structure.), a map to $\mathbb{R}^2$ is clearly insufficient. Using Proposition 77, we can easily deduce more satisfactory statement. Namely, the following can be shown by a straightforward inductive argument.

Proposition 84. *Any trivalent graph Γ has a stabilization Γ' which is also trivalent, such that there is an embedding $h : \Gamma' \to \mathbb{R}^n$ for some n, and (Γ', h) is a smoothable tropical curve.* $\square$

7.6. **An extension of the well-spacedness condition.** Here we give an extension of the well-spacedness condition (Definitions 44, 51) for general curves which have unique bouquet L. *We assume Assumption C described at the begining of Section 6 for simplicity.*

Let (Γ, h) be such a tropical curve. Let H be the space of dual obstruction vectors as usual. Let

$$L = \bigcup_m l_m$$

be the decomposition of the bouquet of Γ into the union of segments, as in the beginning of Subsection 7.1. As before, let $\mathcal{B}_L$ be the set of connected components of $\Gamma \setminus L$.

Take any element $\mathfrak{a} \in H$. Let

$$Supp(\mathfrak{a}) = \bigcup_i l_{\mathfrak{a},i}$$

be the decomposition of $Supp(\mathfrak{a})$ into the union of segments, induced from the decomposition of L above. Let $u_{\mathfrak{a},i}$ be the element of $(N_{\mathbb{C}})^{\vee}$ attached to the component $l_{\mathfrak{a},i}$ by $\mathfrak{a}$. It defines the annihilated affine hyperplane $L_{\mathfrak{a},i} = (u_{\mathfrak{a},i})^{\perp}$ containing $l_{\mathfrak{a},i}$. Let $\{B_{\mathfrak{a},i;k}\}_k$ be the set of elements of $\mathcal{B}_L$ whose closures intersect $l_{\mathfrak{a},i}$.

The intersection of $L_{\mathfrak{a},i}$ and $B_{\mathfrak{a},i;k}$ may have several connected components, but there is the unique one, $B^{cl}_{\mathfrak{a},i;k}$, closest to L. The component $B^{cl}_{\mathfrak{a},i;k}$ is a tree which may or may not have one-valent vertices (when there is no one-valent vertex, then $B_{\mathfrak{a},i;k} = B^{cl}_{\mathfrak{a},i;k}$). Let $d_{\mathfrak{a},i;k}$ be the minimum of the length in the sense of Definition 38 from these one-valent vertices to L. When there is no one-valent vertex, then set $d_{\mathfrak{a},i;k} = \infty$.

Using these notations, we can describe the extended well-spacedness condition.

Definition 85. The tropical curve (Γ, h) satisfying Assumption C is called *well-spaced* if the following condition is satisfied for any $\mathfrak{a} \in H$:

In the set

$$\{d_{\mathfrak{a},i;k}\}_{i,k}$$

of integers $(\cup\infty)$, one of the followings is satisfied.

(1) The minimum is taken by at least two elements.
(2) The minimum is taken by a unique element, the corresponding vertex is four-valent and the tropical curve is locally isomorphic to Example 1 or Example 2 (b) of Subsection 5.2 around this vertex.

The following is proved as in the genus one case (Theorems 45, 52).

Proposition 86. *Let (Γ, h) be a tropical curve as above. Then it is smoothable if and only if it is well-spaced.* □

The well-spacedness condition can be generalized to more general situations, and the moduli space of smoothable pre-log curve can be studied in detail in many cases. We leave the details of these studies to future research.

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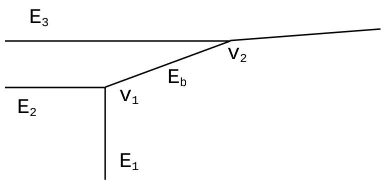

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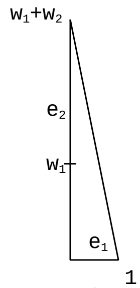

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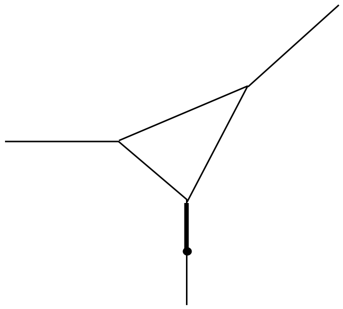

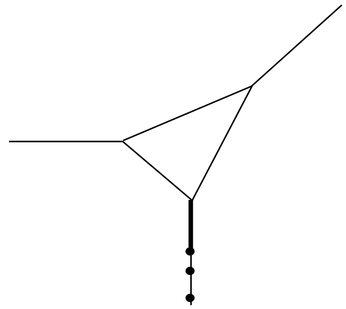

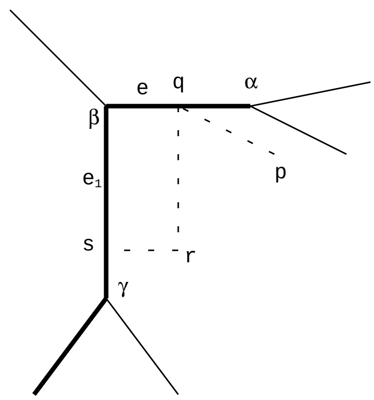

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